ON MOD p ORIENTATION CHARACTERS

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Abstract. Suppose F is a finite extension of $\mathbb{Q}_p$ and G the group of F-points of a connected reductive F-group. We prove that the mod p orientation character of G is trivial, giving a different proof of a result of Schneider–Sorensen.

1. Introduction

Let F denote a finite extension of $\mathbb{Q}_p$ and $\mathbf{G}$ a connected reductive group over F, with $G := \mathbf{G}(F)$ its group of F-rational points. We will often conflate algebraic groups over F with their groups of $\overline{F}$-points. We fix a chamber $\mathcal{C}$ in the semisimple Bruhat–Tits building of G, let I denote the associated Iwahori subgroup and I_1 its pro-unipotent radical, an open pro-p subgroup of G. We assume throughout that I_1 is torsion-free[1]. This assumption guarantees that the cohomological dimension of I_1 is finite, equal to $d := \dim_{\mathbb{Q}_p}(G)$ (see [Ser65, Cor. (1)]).

Given a point x in the semisimple Bruhat–Tits building and a real number $r \geq 0$, we denote by $G_{x,r}$ the depth-r Moy–Prasad subgroup. Our goal will be to prove the following:

Theorem 1.1. *Assume I_1 is torsion-free. Let $r' > 0$, and suppose $g \in G$ satisfies $g \cdot x = x$. Then the adjoint action of g on $\mathrm{H}^d(G_{x,r'}, \mathbb{F}_p)$ is trivial.*

2. Preparation

In order to prove the theorem, we make a few reductions and preparations. Firstly, replacing $\mathbf{G}$ by the Weil restriction $\mathrm{Res}_{F/\mathbb{Q}_p}(\mathbf{G})$, we may assume $F = \mathbb{Q}_p$.

Next, we recall some properties of Moy–Prasad filtrations [MP96]. For each point x in the semisimple Bruhat–Tits building and each real number $r \geq 0$, we let $G_{x,r}$ denote the depth-r Moy–Prasad subgroup. Further, we let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$, and let $\mathfrak{g}_{x,r}$ denote the depth-r Moy–Prasad Lie sublattice. It is a free $\mathbb{Z}_p$-module whose $\mathbb{Q}_p$-span is $\mathfrak{g}(\mathbb{Q}_p)$. These filtrations satisfy the following properties (see [MP96, §§ 3.2, 3.3]):

- If $r \leq s$, then $G_{x,s}$ is a normal subgroup of $G_{x,r}$.
- The set $\{G_{x,r}\}_{r\geq 0}$ gives a neighborhood basis of the identity.
- If x is contained in the closure of the facet containing y, then $G_{x,0+} \leq G_{y,0+}$, where $G_{z,0+} := \bigcup_{s>0} G_{z,s}$ (cf. [DeB02, Def. 3.2.18, Cor. 3.2.19]).
- For $g \in G$, we have $\mathrm{Ad}(g)(G_{x,r}) = G_{g\cdot x,r}$ and $\mathrm{Ad}(g)(\mathfrak{g}_{x,r}) = \mathfrak{g}_{g\cdot x,r}$. In particular, if $g \cdot x = x$, then $\mathrm{Ad}(g)$ preserves $G_{x,r}$ and $\mathfrak{g}_{x,r}$.
- We have $p\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$.

Let $r > 0$ and $y \in \mathcal{C}$, and note that $I_1 = G_{y,0+}$. Since every point x is G-conjugate to a point in $\overline{\mathcal{C}}$, we have

$$\mathrm{Ad}(g)(G_{x,r}) = G_{g\cdot x,r} \leq G_{g\cdot x,0+} \leq G_{y,0+}$$

for some $g \in G$ satisfying $g \cdot x \in \overline{\mathcal{C}}$, by the third and fourth points above. Consequently, we see that the torsion-freeness assumption on I_1 implies that every $G_{x,r}$ is torsion-free (for $r > 0$). Thus, by [Ser65, Cor. (1)] and [Laz65, Thm. V.2.5.8], $G_{x,r}$ is a Poincaré group of dimension d. In particular, we have $\mathrm{H}^d(G_{x,r}, \mathbb{F}_p) \cong \mathbb{F}_p$ (see *op. cit.*, §V.2.5.7), and Theorem 1.1 is trivially true for $p = 2$. Therefore, we may assume that p **is odd**.

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[1] This condition is satisfied if p is sufficiently large relative to the rank of $\mathbf{G}$ and the absolute ramification index of F; for an explicit bound (at least when $\mathbf{G}$ is semisimple), see [Tot99, Prop. 12.1].

Lemma 2.1. *Suppose $r \geq 1$. Then there exists an isomorphism*

$$G_{x,r}/G_{x,r+1} \xrightarrow{\sim} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1} = \mathfrak{g}_{x,r} \otimes_{\mathbb{Z}_p} \mathbb{F}_p.$$

Moreover, if $g \in G$ satisfies $g \cdot x = x$, then this isomorphism is equivariant for the adjoint action of g.

Proof. The claimed isomorphism is known as the *Moy–Prasad isomorphism* and is well-known when $\mathbf{G}$ splits over a tamely ramified extension [Yu01, Cor. 2.4]. When $\mathbf{G}$ is wild, the Moy–Prasad filtration has many deficiencies [Yu02, §§ 0.4, 4.6]. Nonetheless, something can still be said in general. The theory of dilatation, a close relative of the Moy–Prasad filtration, constructs a congruence filtration $(\mathbf{H}_n)_{n\geq 0}$ on any flat affine group scheme $\mathbf{H}$ of finite type over $\mathbb{Z}_p$. The dilatation $\mathbf{H}_n$ is again a flat, affine, finite-type $\mathbb{Z}_p$-group scheme and $\mathbf{H}_n(\mathbb{Z}_p) = \ker(\mathbf{H}(\mathbb{Z}_p) \longrightarrow \mathbf{H}(\mathbb{Z}/p^n\mathbb{Z}))$. In this setting one can prove that when $\mathbf{H}$ is in addition smooth and with connected generic fiber, there is a functorial isomorphism of $\mathbb{Z}/p^n\mathbb{Z}$-group schemes

$$\mathbf{H}_{n,\mathbb{Z}/p^n\mathbb{Z}} \xrightarrow{\sim} \mathfrak{h}_{n,\mathbb{Z}/p^n\mathbb{Z}},$$

where $\mathfrak{h}_n$ is the Lie algebra of $\mathbf{H}_n$ and the subscript $\mathbb{Z}/p^n\mathbb{Z}$ denotes base change. In a forthcoming book on the Bruhat–Tits building, Kaletha and Prasad explain this general result on dilatation [KP, Prop. A.5.17] and use it to prove an even stronger version of the claimed isomorphism ([KP, Thm. 12.4.1]).[2] □

Corollary 2.2. *Suppose p is odd. For $r \geq 1$, we have*

$$\mathrm{H}^1(G_{x,r}, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}, \mathbb{F}_p) = (\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1})^*.$$

Proof. Since $G_{x,r}$ acts trivially on $\mathbb{F}_p$, we have

$$\mathrm{H}^1(G_{x,r}, \mathbb{F}_p) = \mathrm{Hom}^{\mathrm{cts}}(G_{x,r}, \mathbb{F}_p) = \mathrm{Hom}\big((G_{x,r})_\Phi, \mathbb{F}_p\big),$$

where $(G_{x,r})_\Phi := G_{x,r}/\overline{G_{x,r}^p[G_{x,r}, G_{x,r}]}$ denotes the Frattini quotient. The group $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}$ is abelian and p-torsion, and the above lemma implies that the same is true for $G_{x,r}/G_{x,r+1}$. The universal property of the Frattini quotient then gives a surjective map

$$(G_{x,r})_\Phi \twoheadrightarrow G_{x,r}/G_{x,r+1} \xrightarrow{\sim} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}.$$

By dualizing, we obtain

$$\mathrm{Hom}_{\mathbb{F}_p}(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}, \mathbb{F}_p) \hookrightarrow \mathrm{Hom}\big((G_{x,r})_\Phi, \mathbb{F}_p\big) = \mathrm{H}^1(G_{x,r}, \mathbb{F}_p), \tag{1}$$

which implies $\dim_{\mathbb{F}_p}(\mathrm{H}^1(G_{x,r}, \mathbb{F}_p)) \geq \dim_{\mathbb{F}_p}(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1}) = \dim_{\mathbb{Q}_p}(G) = d$. On the other hand, [Klo11, Prop. 1.2] implies that $\dim_{\mathbb{F}_p}(\mathrm{H}^1(G_{x,r}, \mathbb{F}_p)) \leq d$. Thus, the injection (1) is an isomorphism. □

Corollary 2.3.
(a) If $p \geq 5$, then $G_{x,r}$ is a uniform pro-p group for all $r \geq 1$.
(b) If $p = 3$, then $G_{x,r}$ is a uniform pro-3 group for all $r \gg 0$ sufficiently large.

Proof. For $p \geq 5$, this follows from the previous corollary and [KS14, Prop. 1.10, Rmk. 1.11]. Assume $p = 3$. By [DdSMS99, Cor. 4.3], the group I_1 possesses an open normal subgroup H which is uniform, and in particular saturable. Since the Moy–Prasad subgroups give a neighborhood basis of the identity, there exists $r_0 > 0$ such that $G_{x,r_0} \leq H$. Then, for any $r \geq \max\{r_0, 1\}$ we have $G_{x,r} \leq H$, and the result follows from the previous corollary and [KS14, Cor. 1.7]. □

3. Proof

We now prove Theorem 1.1, given the reductions in the previous section. Thus, we assume that p is odd, $\mathbf{G}$ is defined over $\mathbb{Q}_p$, I_1 is torsion-free, and $g \in G$ satisfies $g \cdot x = x$.

By Corollary 2.3, there exists a real number $r \geq \max\{r', 1\}$ for which the group $G_{x,r}$ is uniform. By [Ser02, pf. of Prop. 30], the corestriction map

$$\mathrm{cor} : \mathrm{H}^d(G_{x,r}, \mathbb{F}_p) \longrightarrow \mathrm{H}^d(G_{x,r'}, \mathbb{F}_p)$$

is an isomorphism, which is furthermore equivariant for the adjoint action of g. Therefore, it suffices to compute the action of $\mathrm{Ad}(g)$ on $\mathrm{H}^d(G_{x,r}, \mathbb{F}_p)$.

[2] We cite [KP] with the caveat that it is still a draft and is undergoing proof-reading and revisions.

Since $G_{x,r}$ is uniform, [SW00, Thm. 5.1.5] and Corollary 2.2 imply that we have $\mathrm{Ad}(g)$-equivariant isomorphisms of $\mathbb{F}_p$-vector spaces

$$\mathrm{H}^d(G_{x,r}, \mathbb{F}_p) \cong \bigwedge\nolimits^d_{\mathbb{F}_p} \mathrm{H}^1(G_{x,r}, \mathbb{F}_p) \cong \bigwedge\nolimits^d_{\mathbb{F}_p} (\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1})^*.$$

Therefore, by dualizing, it suffices to prove that $\mathrm{Ad}(g)$ acts trivially on

$$\bigwedge\nolimits^d_{\mathbb{F}_p} \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+1} \cong \bigwedge\nolimits^d_{\mathbb{F}_p} \left(\mathfrak{g}_{x,r} \otimes_{\mathbb{Z}_p} \mathbb{F}_p\right) \cong \left(\bigwedge\nolimits^d_{\mathbb{Z}_p} \mathfrak{g}_{x,r}\right) \otimes_{\mathbb{Z}_p} \mathbb{F}_p.$$

Using the above isomorphisms, it suffices to show that $\bigwedge^d \mathrm{Ad}(g)$ acts trivially on $\bigwedge^d_{\mathbb{Z}_p} \mathfrak{g}_{x,r}$. The latter is a free $\mathbb{Z}_p$-module of rank 1 since $\mathfrak{g}_{x,r}$ is a free $\mathbb{Z}_p$-module of rank d, and we are reduced to showing that $\det_{\mathbb{Z}_p}(\mathrm{Ad}(g)) = 1$ for the adjoint action of g on $\mathfrak{g}_{x,r}$ (note that the determinant is defined as $\mathfrak{g}_{x,r}$ is free of finite rank).

In order to prove that the determinant of the map $\mathrm{Ad}(g) : \mathfrak{g}_{x,r} \longrightarrow \mathfrak{g}_{x,r}$ is equal to 1, it suffices to do so after tensoring by $\overline{\mathbb{Q}}_p$:

$$\mathrm{Ad}(g) : \mathfrak{g} = \mathfrak{g}_{x,r} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \longrightarrow \mathfrak{g} = \mathfrak{g}_{x,r} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p.$$

Recall that the group $\mathbf{G}$ has an almost-direct product decomposition $\mathbf{G} = \mathbf{Z}\mathbf{G}^{\mathrm{der}}$, where $\mathbf{Z}$ denotes the connected center of $\mathbf{G}$ and $\mathbf{G}^{\mathrm{der}}$ denotes the derived subgroup (see [Spr09, Cor. 8.1.6(i)]). Let us write $g = zg'$ with $z \in \mathbf{Z}$ and $g' \in \mathbf{G}^{\mathrm{der}}$. As linear maps on $\mathfrak{g}$, we have

$$\mathrm{Ad}(g) = \mathrm{Ad}(z) \circ \mathrm{Ad}(g') = \mathrm{Ad}(g'),$$

since z is central. Therefore, we get

$$\det_{\overline{\mathbb{Q}}_p}(\mathrm{Ad}(g)) = \det_{\overline{\mathbb{Q}}_p}(\mathrm{Ad}(g')) = 1,$$

since $\mathbf{G}^{\mathrm{der}}$ is generated by commutators. This concludes the proof of Theorem 1.1.

4. Orientation character

We once again make no assumption on p, and continue to assume I_1 is torsion-free. We define the mod p orientation character $\xi : G \longrightarrow \mathbb{F}_p^\times$ as follows. Let $\mathbb{F}_p$ denote the trivial G-representation. For $g \in G$, we set $I_g := I_1 \cap gI_1g^{-1}$, and let $\xi(g)$ denote the scalar defined by the sequence of isomorphisms

$$\mathbb{F}_p \xrightarrow{\mathrm{tr}^{-1}_{I_{g^{-1}}}} \mathrm{H}^d(I_{g^{-1}}, \mathbb{F}_p) \xrightarrow{g_*} \mathrm{H}^d(I_g, \mathbb{F}_p) \xrightarrow{\mathrm{tr}_{I_g}} \mathbb{F}_p.$$

Here $\mathrm{tr}_K : \mathrm{H}^d(K, \mathbb{F}_p) \xrightarrow{\sim} \mathbb{F}_p$ is the trace map, defined in [Ser02, §I.4.5, Prop. 30(b)] for any torsion-free open pro-p subgroup of G. It is known that the association $g \longmapsto \xi(g)$ gives a homomorphism, which is trivial on I_1 ([SS] and [Koz, Lem. 4.11, Def. 4.12]).

The following result was first proved by Schneider–Sorensen.

Theorem 4.1. *The character $\xi : G \longrightarrow \mathbb{F}_p^\times$ is trivial.*

Proof. We fix an apartment of the semisimple Bruhat–Tits building which contains $\mathcal{C}$, and let $\mathbf{S}$ denote the associated maximal F-split torus. Recall that the group G has a Bruhat factorization

$$G = I_1 N I_1,$$

where N denotes the group of F-points of the normalizer of $\mathbf{S}$ (see [Vig16, Prop. 3.35]). Since ξ is trivial on I_1, it suffices to prove that it is also trivial on N.

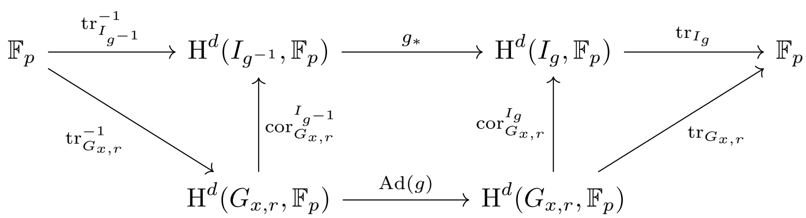
The group N acts by simplicial automorphisms on the apartment associated to $\mathbf{S}$, and we define $N_{\mathcal{C}}$ to be the subgroup of N which stabilizes $\mathcal{C}$. By equations (53) of [Vig16], the group N is generated by $N_{\mathcal{C}}$ and representatives of affine reflections in the walls of $\mathcal{C}$. It therefore suffices to prove that ξ is trivial on $N_{\mathcal{C}}$ and on each such affine reflection.

Suppose first that $g \in N_{\mathcal{C}}$, so that g normalizes I_1 and $I_g = I_{g^{-1}} = I_1$. Let x denote the barycenter of $\mathcal{C}$, so that $g \cdot x = x$ and $I_1 = G_{x,r}$ for some sufficiently small $r > 0$. By Theorem 1.1, we have that

$$g_* = \mathrm{Ad}(g) : \mathrm{H}^d(I_1, \mathbb{F}_p) \longrightarrow \mathrm{H}^d(I_1, \mathbb{F}_p)$$

is the identity map. By definition of ξ, we get $\xi(g) = 1$.

Suppose now that g is a representative of an affine reflection in a wall of $\mathcal{C}$. Let $x \in \overline{\mathcal{C}}$ denote a point on this wall, so that $g \cdot x = x$. Since $I_g \cap I_{g^{-1}}$ is open in G, we may choose $r > 0$ such that $G_{x,r} \leq I_g \cap I_{g^{-1}}$. By canonicity of the trace map (cf. [Koz, Lem. 4.9]) and functoriality of the corestriction map, the following diagram is commutative:



(Recall that by [Ser02, pf. of Prop. 30], the displayed corestriction maps are isomorphisms.) By Theorem 1.1, the bottom horizontal arrow is the identity map, which implies $\xi(g) = 1$. □

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