DG HECKE MODULES

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Abstract. These are some notes about Schneider's derived equivalence of categories. They follow Schneider's article closely.

1. Motivation

Let F denote a finite extension of $\mathbb{Q}_p$, k an arbitrary field, and $\mathbf{G}$ a connected reductive group over F. We set $G := \mathbf{G}(F)$, and let I denote a pro-p-Iwahori subgroup of G. We also let

$$X := \text{c-ind}_I^G(k) = \left\{ f : G \longrightarrow k : \begin{array}{l} \diamond\ f(gi) = f(g)\ \forall g \in G, \forall i \in I, \\ \diamond\ f \text{ has compact support.} \end{array} \right\},$$

and let

$$\mathcal{H} := \mathrm{End}_G(X)^{\mathrm{op}}$$

denote the opposite algebra of G-equivariant endomorphisms of X.

We have the following results on the relationship between representations of G and (left) $\mathcal{H}$-modules:

(1) (Borel, Bernstein) When $k = \mathbb{C}$, we have an equivalence of categories

$$\begin{array}{rcl} \mathfrak{Mod}_{\mathbb{C}}^I(G) & \cong & \mathcal{H}\text{-}\mathfrak{Mod} \\ V & \longmapsto & V^I \cong \mathrm{Hom}_I(\mathbb{C}, V|_I) \cong \mathrm{Hom}_G(X, V). \end{array}$$

(2) (Ollivier, K.) When $k = \overline{\mathbb{F}}_p$, and G is a torus, $\mathrm{GL}_2(\mathbb{Q}_p)$ or $\mathrm{SL}_2(\mathbb{Q}_p)$, we have an equivalence of categories

$$\begin{array}{rcl} \mathfrak{Mod}_{\overline{\mathbb{F}}_p}^I(G) & \cong & \mathcal{H}\text{-}\mathfrak{Mod} \\ V & \longmapsto & V^I \cong \mathrm{Hom}_I(\overline{\mathbb{F}}_p, V|_I) \cong \mathrm{Hom}_G(X, V). \end{array}$$

This equivalence is also known to fail in certain cases (e.g., for $\mathrm{GL}_2(F)$, F an unramified extension of $\mathbb{Q}_p$).

Our goal will be to remedy the second situation by passing to derived categories: by deriving the Hom functor, we recover a (derived) equivalence, and obtain modules over a certain differential graded Hecke algebra.

2. $\mathfrak{Mod}_k(G)$ and its derived category

We assume from this point onwards that $\mathrm{char}(k) = p$, and that the group I is *torsion-free* (we will mention below where this assumption is necessary). We first collect some important properties of the category $\mathfrak{Mod}_k(G)$ of smooth k-representations of G.

Proposition 2.1. *The category $\mathfrak{Mod}_k(G)$ is a Grothendieck abelian category which admits arbitrary limits and colimits.*

Recall that an abelian category $\mathcal{A}$ is called *Grothendieck* if it admits arbitrary colimts, filtered colimits are exact, and it possesses a generator. A generator is an object Y which "detects isomorphisms:" a map $f : Z \longrightarrow W$ in $\mathcal{A}$ is an isomorphism if and only if the induced map $f_* : \mathrm{Hom}_{\mathcal{A}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, W)$ is bijective.

Remark 2.2. The Grothendieck property implies that $\mathfrak{Mod}_k(G)$ has enough injectives.

We now consider the left-exact functor of I-invariants $\mathrm{H}^0(I, -) \simeq \mathrm{Hom}_G(X, -)$:

$$\begin{array}{rcl} \mathfrak{Mod}_k(G) & \longrightarrow & k\text{-}\mathfrak{Vec} \\ V & \longmapsto & V^I \cong \mathrm{Hom}_G(X, V). \end{array}$$

(we will forget about the $\mathcal{H}$-module structure on V^I for now).

Notation. We use $\mathsf{C}(G), \mathsf{K}(G)$, and $\mathsf{D}(G)$ to denote the categories $\mathsf{C}(\mathfrak{Mod}_k(G)), \mathsf{K}(\mathfrak{Mod}_k(G))$, and $\mathsf{D}(\mathfrak{Mod}_k(G))$, respectively.

Since $\mathfrak{Mod}_k(G)$ is Grothendieck abelian, the total derived functor $\mathrm{RH}^0(I, -) \simeq \mathrm{RHom}_G(X, -)$ exists:

$$\begin{array}{rcl} \mathsf{D}(G) & \longrightarrow & \mathsf{D}(k\text{-}\mathfrak{Vec}) \\ V^\bullet & \longmapsto & \mathrm{RH}^0(I, V^\bullet) = \mathrm{RHom}_G(X, V^\bullet). \end{array}$$

In order to describe $\mathrm{RH}^0(I, -)$ more explicitly, we recall the following construction (which will be used later): Let $A^\bullet, B^\bullet$ be two complexes in $\mathsf{C}(G)$ with differentials $\mathrm{d}_A^\bullet$ and $\mathrm{d}_B^\bullet$, respectively. We define the *Hom complex* $\mathrm{Hom}_{\mathcal{A}}^\bullet(A^\bullet, B^\bullet) \in \mathsf{C}(k\text{-}\mathfrak{Vec})$ as follows:

$$\begin{array}{rcl} \mathrm{Hom}_G^n(A^\bullet, B^\bullet) & := & \displaystyle\prod_{i \in \mathbb{Z}} \mathrm{Hom}_G(A^i, B^{i+n}) \\ \mathrm{d}^n\left((\varphi^i)_i\right)(a_j) & := & \mathrm{d}_B^{j+n}\left(\varphi^j(a_j)\right) - (-1)^n \varphi^{j+1}\left(\mathrm{d}_A^j(a_j)\right). \end{array}$$

Note that we don't impose any commutation conditions on the maps $(\varphi^i)_i$. This gives a bifunctor $\mathrm{Hom}_G^\bullet(-,-) : \mathsf{C}(G)^{\mathrm{op}} \times \mathsf{C}(G) \longrightarrow \mathsf{C}(k\text{-}\mathfrak{Vec})$, which descends to homotopy categories. (Note that this is simply the extension of the Hom bifunctor $\mathrm{Hom}_G(-,-) : \mathfrak{Mod}_k(G)^{\mathrm{op}} \times \mathfrak{Mod}_k(G) \longrightarrow k\text{-}\mathfrak{Vec}$ to the category of complexes, using the product totalization.)

The following properties are easily checked.

$$\begin{array}{rcl} \mathrm{Hom}_G^n(A^\bullet, B^\bullet) & = & \mathrm{Hom}_G^0(A^\bullet, B[n]^\bullet), \\ \mathrm{Z}^0(\mathrm{Hom}_G^\bullet(A^\bullet, B^\bullet)) & = & \mathrm{Hom}_{\mathsf{C}(G)}(A^\bullet, B^\bullet), \\ \mathrm{H}^0(\mathrm{Hom}_G^\bullet(A^\bullet, B^\bullet)) & = & \mathrm{Hom}_{\mathsf{K}(G)}(A^\bullet, B^\bullet), \\ \mathrm{H}^n(\mathrm{Hom}_G^\bullet(A^\bullet, B^\bullet)) & = & \mathrm{Hom}_{\mathsf{K}(G)}(A^\bullet, B[n]^\bullet). \end{array}$$

We recall that a complex $A^\bullet$ is called K*-injective* (or *homotopically injective*) if

$$B^\bullet \text{ is exact} \Longrightarrow \mathrm{Hom}_G^\bullet(B^\bullet, A^\bullet) \text{ is exact.}$$

We denote the additive subcategory of K-injective complexes in $\mathsf{K}(G)$ by $\mathsf{K}_{\mathrm{inj}}(G)$. Some useful properties of this notion (most of these follow from the fact that $\mathfrak{Mod}_k(G)$ is a Grothendieck abelian category):

(1) If $J^\bullet \in \mathsf{C}(G)$ is a bounded below complex of injective objects, then $J^\bullet$ is K-injective.
(2) The category $\mathfrak{Mod}_k(G)$ has enough K-injectives; that is, if $A^\bullet$ is any complex in $\mathsf{C}(G)$, then there exists a K-injective complex $J^\bullet$ and a quasi-isomorphism

$$A^\bullet \xrightarrow{\mathrm{qis}} J^\bullet.$$

(3) If $A^\bullet \in \mathsf{K}(G)$ and $J^\bullet \in \mathsf{K}_{\mathrm{inj}}(G)$, then
$$\mathrm{Hom}_{\mathsf{K}(G)}(A^\bullet, J^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(G)}(A^\bullet, J^\bullet),$$
where we consider the objects on the right-hand side as elements of the derived category via the localization functor $\mathsf{K}(G) \longrightarrow \mathsf{D}(G)$.

(4) Moreover, the localization functor $\mathsf{K}(G) \longrightarrow \mathsf{D}(G)$ induces an equivalence of triangulated categories
$$\mathsf{K}_{\mathrm{inj}}(G) \xrightarrow{\sim} \mathsf{D}(G).$$
We let $\mathbf{i} : \mathsf{D}(G) \xrightarrow{\sim} \mathsf{K}_{\mathrm{inj}}(G)$ denote a quasi-inverse. Note that, given $V^\bullet \in \mathsf{D}(G)$, the complex $\mathbf{i}(V^\bullet)$ is a K-injective resolution of $V^\bullet$.

(5) Suppose $F : \mathfrak{Mod}_k(G) \longrightarrow k\text{-}\mathfrak{Vec}$ is a left-exact functor. Then $\mathrm{R}F : \mathsf{D}(G) \longrightarrow \mathsf{D}(k\text{-}\mathfrak{Vec})$ exists, and we may compute it as follows:
$$\mathrm{R}F(A^\bullet) = F(\mathbf{i}(A^\bullet)),$$
where $F : \mathsf{K}(G) \longrightarrow \mathsf{K}(k\text{-}\mathfrak{Vec})$ is the natural extension of F to homotopy categories, and where we consider the right-hand side as an element of $\mathsf{D}(k\text{-}\mathfrak{Vec})$ via the canonical localization.

In particular, we see that
$$\begin{tikzcd}
\mathrm{RH}^0(I, V^\bullet) \arrow[r, equal, "\sim"] \arrow[d, equal, "\wr"] & \mathbf{i}(V^\bullet)^I \arrow[d, equal, "\wr"] \\
\mathrm{RHom}_G(X, V^\bullet) \arrow[r, equal, "\sim"] & \mathrm{Hom}^\bullet_G(X, \mathbf{i}(V^\bullet))
\end{tikzcd}$$
where we view X as a complex concentrated in degree 0.

Some useful results:

Lemma 2.3. *The representation $X \in \mathsf{D}(G)$ (in degree 0) is a compact object, that is, $\mathrm{Hom}_{\mathsf{D}(G)}(X, -)$ commutes with arbitrary direct sums.*

Sketch of proof. We have
$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}(G)}(X, V^\bullet) &\cong \mathrm{H}^0\left(\mathrm{RHom}_G(X, V^\bullet)\right) \\
&\cong \mathrm{H}^0\left(\mathrm{RH}^0(I, V^\bullet)\right) \\
&\cong \mathbb{H}^0(I, V^\bullet),
\end{aligned}$$
and one easily checks that the hypercohomology functors $\mathbb{H}^i(I, -)$ commute with arbitrary direct sums (since the regular cohomology functors do). $\square$

The following important proposition will be the key input into the main theorem. Here is where we have to assume that the characteristic of F is 0 and I is torsion-free, so that the completed group ring of I has nice properties.

Proposition 2.4. *Let $V^\bullet \in \mathsf{D}(I)$. Then $V^\bullet = 0$ in $\mathsf{D}(I)$ if and only if $\mathrm{RH}^0(I, V^\bullet) = 0$ in $\mathsf{D}(k\text{-}\mathfrak{Vec})$.*

Sketch of proof. The completed group ring $\Omega := k[\![I]\!]$ is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and any $V \in \mathfrak{Mod}_k(I)$ is naturally a torsion module over Ω, that is,
$$V = \bigcup_{j \geq 0} V^{\mathfrak{m}^j = 0}.$$

We shall use the following short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{\oplus n_j} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0,$$

where the isomorphism follows from the noetherian property of Ω.

Let $V^\bullet \in \mathsf{D}(I)$, and choose a quasi-isomorphism $V^\bullet \xrightarrow{\mathrm{qis}} C^\bullet$ into a complex $C^\bullet$, where each term is $\mathrm{H}^0(I,-)$-acyclic. Taking $\mathrm{Hom}_I(-,C^m)$ of the short exact sequence above and using that C^m is also $\mathrm{Hom}_I(\Omega/\mathfrak{m}^j,-)$-acyclic gives an exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_I(\Omega/\mathfrak{m}^j, C^m) \cong (C^m)^{\mathfrak{m}^j=0} \longrightarrow \mathrm{Hom}_I(\Omega/\mathfrak{m}^{j+1}, C^m) \cong (C^m)^{\mathfrak{m}^{j+1}=0} \\ \longrightarrow \mathrm{H}^0(I,C^m)^{n_j} \cong ((C^m)^I)^{\oplus n_j} \longrightarrow 0\end{aligned}$$

for every $j \geq 0$. Collecting terms, we obtain exactness of

$$(\star) \qquad 0 \longrightarrow \mathrm{Hom}_I^\bullet(\Omega/\mathfrak{m}^j, C^\bullet) \longrightarrow \mathrm{Hom}_I^\bullet(\Omega/\mathfrak{m}^{j+1}, C^\bullet) \longrightarrow ((C^\bullet)^I)^{n_j} \longrightarrow 0.$$

Assume now that $\mathrm{RH}^0(I,V^\bullet) = 0$; this means exactly that the complex $(C^\bullet)^I$ is exact. Taking the long exact sequence associated in cohomology of $(\star)$ and inducting on j gives that the complex $\mathrm{Hom}_I^\bullet(\Omega/\mathfrak{m}^j, C^\bullet) = (C^\bullet)^{\mathfrak{m}^j=0}$ is exact for every j. By smoothness, both $C^\bullet$ and $V^\bullet$ are exact. □

The above proposition implies the following.

Theorem 2.5. *The element X is a compact generator of $\mathsf{D}(G)$: if $\mathcal{C} \subset \mathsf{D}(G)$ is a strictly full (i.e., full and closed under isomorphism) triangulated subcategory, closed under direct sums and containing X, then $\mathcal{C} = \mathsf{D}(G)$.*

Themes of proof. By the above two results, the family of shifts $X[j]$, $j \in \mathbb{Z}$ are compact objects in $\mathsf{D}(G)$, and

$$\mathrm{Hom}_{\mathsf{D}(G)}(X[j], V^\bullet) \cong \mathbb{H}^{-j}(I, V^\bullet).$$

Therefore, the family $\{X[j]\}_{j\in\mathbb{Z}}$ detects whether $V^\bullet$ is 0 in $\mathsf{D}(G)$. The rest follows from properties of triangulated categories. □

3. The algebra $\mathcal{H}^\bullet$ and its derived category

We choose, once an for all, an injective resolution $\mathcal{I}^\bullet$ of X in $\mathsf{C}(G)$:

$$X \xrightarrow{\mathrm{qis}} \mathcal{I}^\bullet$$

We may assume that $\mathcal{I}^i = 0$ for $i < 0$. We define the *differential graded (DG) Hecke algebra* $\mathcal{H}^\bullet$ as

$$\mathcal{H}^\bullet := \mathrm{End}_G^\bullet(\mathcal{I}^\bullet)^{\mathrm{op}} := \mathrm{Hom}_G^\bullet(\mathcal{I}^\bullet, \mathcal{I}^\bullet)^{\mathrm{op}}.$$

We have already defined the differential (which we denote $\mathrm{d}_{\mathcal{H}}^\bullet$), and the product structure is given as follows: given $a = (a^i)_i \in \mathcal{H}^n, b = (b^j)_j \in \mathcal{H}^m$, we set

$$(ba)^\ell = (-1)^{mn} a^{\ell+m} \circ b^\ell$$

(recall that we have defined $\mathcal{H}^\bullet$ with "op").

Remark 3.1. We have

$$\begin{aligned}\mathrm{H}^i(\mathcal{H}^\bullet) &\cong \mathrm{Hom}_{\mathsf{K}(G)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet[i]) \\ &\cong \mathrm{Hom}_{\mathsf{D}(G)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet[i]) \\ &\cong \mathrm{Hom}_{\mathsf{D}(G)}(X, X[i])\end{aligned}$$

$$
\begin{aligned}
&\cong \mathrm{Ext}^i_G(X,X)\\
&\cong \mathrm{H}^i(\mathrm{Hom}_G(X,\mathcal{I}^\bullet))\\
&\cong \mathrm{H}^i((\mathcal{I}^\bullet)^I)\\
&\cong \mathrm{H}^i(I,X).
\end{aligned}
$$

The assumption $\mathrm{char}(F)=0$ implies that I is a p-adic analytic group, and therefore the torsion-free assumption guarantees its cohomological dimension is equal to its dimension as a p-adic analytic group. Therefore, $\mathrm{H}^i(\mathcal{H}^\bullet)=0$ for i outside $[0,\dim(I)]$.

In particular, we have $\mathrm{H}^0(\mathcal{H}^\bullet)\cong \mathrm{H}^0(I,X)\cong X^I=\mathcal{H}$, the "usual" Iwahori–Hecke algebra. One can show (cf. Schneider's article) that, as a right $\mathcal{H}$-module, the top cohomology $\mathrm{H}^{\dim(I)}(I,X)$ takes the form

$$\mathrm{H}^{\dim(I)}(I,X)\cong \mathrm{H}^{\dim(I)}(I,\mathbb{F}_p)\otimes_{\mathbb{F}_p}\mathcal{H}^\tau\cong\mathcal{H}^\tau,$$

where $\mathcal{H}^\tau$ denotes the space $\mathcal{H}$, with a twisted action of $\mathcal{H}$ from the right. The second isomorphism above follows from choosing a basis for the one-dimensional $\mathbb{F}_p$-vector space $\mathrm{H}^{\dim(I)}(I,\mathbb{F}_p)$. Recent work of Ollivier–Schneider also computes (part of) $\mathrm{H}^1(I,X)$, where I is a pro-p Iwahori subgroup of $\mathrm{SL}_2(F)$.

A differential graded (left) $\mathcal{H}^\bullet$-module is a complex $M^\bullet=\bigoplus_{j\in\mathbb{Z}}M^j$ of vector spaces equipped with a differential $\mathrm{d}^\bullet_M$ and a graded action of $\mathcal{H}^\bullet$, which satisfies the Leibniz rule:

$$\mathrm{d}^{n+\ell}_M(am)=\mathrm{d}^n_{\mathcal{H}}(a)m+(-1)^n a\mathrm{d}^\ell_M(m).$$

where $a\in\mathcal{H}^n, m\in M^\ell$. We define morphisms of DG modules in the obvious way, and we may then form the homotopy and derived categories of DG $\mathcal{H}^\bullet$-modules, denoted $\mathsf{K}(\mathcal{H}^\bullet)$ and $\mathsf{D}(\mathcal{H}^\bullet)$, respectively.

Remark 3.2. The algebra $\mathcal{H}^\bullet$ depends on the choice of the injective resolution $\mathcal{I}^\bullet$ of X, but it is well-defined *up to quasi-isomorphism*. In other words, any other choice of injective resolution will give a quasi-isomorphic algebra. Moreover, any other choice of injective resolution will yield an equivalent derived category, so $\mathsf{D}(\mathcal{H}^\bullet)$ is well-defined up to equivalence.

Consider now the following functors:

$$
\begin{array}{rcl}
\mathsf{D}(G) & \underset{T}{\overset{H}{\rightleftarrows}} & \mathsf{D}(\mathcal{H}^\bullet)\\
V^\bullet & \longmapsto & \mathrm{Hom}^\bullet_G(\mathcal{I}^\bullet,\mathbf{i}(V^\bullet))\\
\mathcal{I}^\bullet\otimes_{\mathcal{H}^\bullet}\mathbf{p}(M^\bullet) & \longleftarrow\!\shortmid & M^\bullet
\end{array}
$$

(Here, one uses K-*projective* resolutions (defined in the obvious way) to define the total tensor product. The category $\mathsf{K}(\mathcal{H}^\bullet)$ has enough K-projectives, and $\mathsf{K}_{\mathrm{pro}}(\mathcal{H}^\bullet)$ is equivalent to the derived category $\mathsf{D}(\mathcal{H}^\bullet)$ by the localization functor. We let $\mathbf{p}:\mathsf{D}(\mathcal{H}^\bullet)\xrightarrow{\sim}\mathsf{K}_{\mathrm{pro}}(\mathcal{H}^\bullet)$ denote a quasi-inverse.) Composing the functor H with the forgetful functor $\mathsf{D}(\mathcal{H}^\bullet)\longrightarrow\mathsf{D}(k\text{-}\mathfrak{Vec})$ gives a functor which is naturally isomorphic to $\mathrm{RH}^0(I,-)$. Moreover, one easily sees that T is left adjoint to H:

$$\mathrm{Hom}_{\mathsf{D}(G)}(\mathcal{I}^\bullet\otimes_{\mathcal{H}^\bullet}\mathbf{p}(M^\bullet),V^\bullet)\cong\mathrm{Hom}_{\mathsf{D}(\mathcal{H}^\bullet)}(M^\bullet,\mathrm{Hom}^\bullet_G(\mathcal{I}^\bullet,\mathbf{i}(V^\bullet)))$$

The main theorem is now the following:

Theorem 3.3. *The functor*

$$H : \mathsf{D}(G) \longrightarrow \mathsf{D}(\mathcal{H}^\bullet)$$

is an equivalence of triangulated categories.

Idea of proof. This follows from a general theorem of Keller. The relevant fact is that the free module $\mathcal{H}^\bullet$ is a compact generator of $\mathsf{D}(\mathcal{H}^\bullet)$, and the functor H sends X to $\mathcal{H}^\bullet$, thus identifying the two categories. $\square$

Remarks. (1) The subcategory $\mathsf{D}(G)^c$ consisting of compact objects is an analog of the subcategory of perfect complexes in the derived category of a ring. It is the smallest strictly full subcategory closed under direct summands and containing X.

(2) The assumption that I has finite cohomological dimension implies that H resticts to a fully faithful functor between bounded derived categories:

$$H : \mathsf{D}^{\mathrm{b}}(G) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathcal{H}^\bullet).$$

However, the Eilenberg–Moore spectral sequence shows that T does not preserve bounded derived categories in general, so we do not get an equivalence on the bounded subcategories.

4. The example of $\mathbb{Z}_p$

All of the results above hold in the more general setting of G being a p-adic Lie group, and I a torsion-free pro-p subgroup. In this example, we take $G = I = \mathbb{Z}_p$, so that I has cohomological dimension 1. The category $\mathfrak{Mod}_k(G)$ of smooth G-representations is then equivalent to the category $\Omega\text{-}\mathfrak{Mod}^{\mathrm{tors}}$ of torsion Ω-modules, where $\Omega = k[\![I]\!] \cong k[\![t]\!]$ denotes the completed group ring of I. Here, a topological generator of I is sent to $t+1$.

In this case, $X = k$ is the trivial G-module, which corresponds to the Ω-module $\Omega/t \cong k$. Moreover, we can compute an explicit injective resolution of X:

$$0 \longrightarrow k \longrightarrow C^\infty(G,k) \longrightarrow C^\infty(G,k) \longrightarrow 0 \longrightarrow \dots$$

We have $\mathrm{End}_G(C^\infty(G,k)) \cong \Omega$, and therefore $\mathcal{H}^\bullet$ is concentrated in degrees $-1, 0$, and 1, and takes the form

$$\dots \longrightarrow k[\![t]\!] \overset{a\mapsto(ta,ta)}{\longrightarrow} k[\![t]\!] \oplus k[\![t]\!] \overset{(a,b)\mapsto t(a-b)}{\longrightarrow} k[\![t]\!] \dots$$

The product is given by

$$(a_{-1},\ (a_0,b_0),\ a_1)\cdot(a'_{-1},\ (a'_0,b'_0),\ a'_1) = (a_{-1}a'_0+b_0a'_{-1},\ (a_0a'_0-a_1a'_{-1}, b_0b'_0-a_{-1}a'_1),\ a_0a'_1+a_1b'_0).$$

A more convenient way of thinking about the algebra is $\mathcal{H}^\bullet$ is by considering its cohomology algebra: we have $\mathrm{H}^0(I,k) = k$, $\mathrm{H}^1(I,k) = k$, with 0 differential. This algebra is isomorphic to $k[\varepsilon]/\varepsilon^2$, with k in degree 0 and $k\varepsilon$ in degree 1. More importantly, the map including k diagonally in $\mathcal{H}^0$ and $\mathcal{H}^1$ gives a quasi-isomorphism between $\mathcal{H}^\bullet$ and its cohomology algebra; that is, $\mathcal{H}^\bullet$ is *formal*. Note that formality does not hold in general. We therefore obtain an equivalence

$$\mathsf{D}(\mathbb{Z}_p) \cong \mathsf{D}(k[\varepsilon]/\varepsilon^2).$$

Based on the sign conventions chosen for left DG modules, a DG module over $k[\varepsilon]/\varepsilon^2$ is a graded k vector space with two anti-commuting differentials of degree 1. Explicitly, let $M^\bullet = \bigoplus_{j\in\mathbb{Z}} M^j$ be a DG module over $k[\varepsilon]/\varepsilon^2$, with differential $\mathrm{d}_M^\bullet$, and let $m \in M^j$. Then

$$\mathrm{d}_M^{j+1}(\varepsilon\cdot m) = \mathrm{d}_{\mathcal{A}}^1(\varepsilon)\cdot m + (-1)^{\deg(\varepsilon)}\varepsilon\cdot \mathrm{d}_M^j(m) = -\varepsilon\cdot \mathrm{d}_M^j(m),$$

that is, $\mathrm{d}_M^{j+1} \circ \varepsilon = -\varepsilon \circ \mathrm{d}_M^j$. Given a smooth G-representation, we can naturally form a graded $k[\varepsilon]/\varepsilon^2$-module $k[\varepsilon]/\varepsilon^2 \otimes_k V$ in degrees 0 and 1, with differential

$$\mathrm{d}_V^\bullet(v_0 + v_1\varepsilon) = ((\gamma - 1).v_0)\,\varepsilon,$$

where γ denotes a topological generator of G, chosen as in the isomorphism $k[\![I]\!] \cong k[\![t]\!]$. That is,

$$(k[\varepsilon]/\varepsilon^2 \otimes_k V, \mathrm{d}_V^\bullet) = \ldots \longrightarrow 0 \longrightarrow V \xrightarrow{v \mapsto (\gamma-1).v} V \longrightarrow 0 \longrightarrow \ldots$$

We then have that the equivalence above sends

$$\mathsf{D}(G) \supset \mathfrak{Mod}_k(G) \ni V \longmapsto (k[\varepsilon]/\varepsilon^2 \otimes_k V, \mathrm{d}_V^\bullet) \in \mathsf{D}(k[\varepsilon]/\varepsilon^2).$$

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