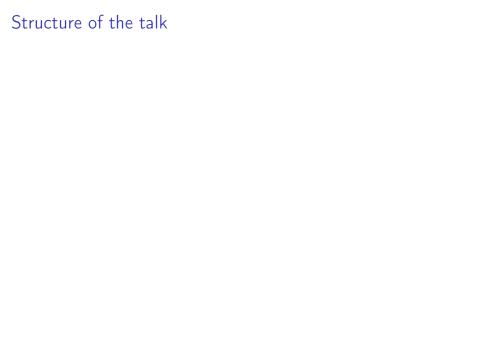
Mod p points on Shimura varieties of parahoric level

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May 30 2020



Structure of the talk

Introduction to the Langlands-Rapoport conjecture and a quick survey of previous work

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Idea of the proof



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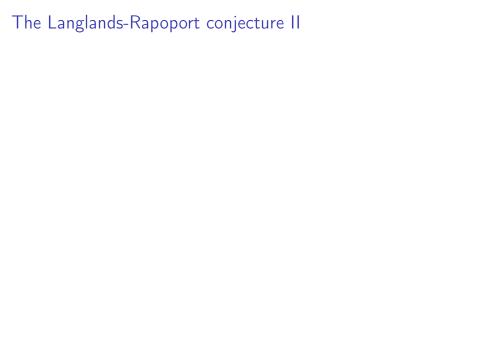
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Understanding these integral models has interesting applications, e.g. construction of Galois representations (Deligne, Langlands), Ribet's proof of the ϵ -conjecture.



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For example $G=\operatorname{GL}_2,X=\mathbb{H}^\pm$ and $K_p=\operatorname{GL}_2(\mathbb{Z}_p)$ or $K_p=\Gamma_0(p)$, then $E=\mathbb{Q}$ and the integral models from the previous slide are 'good'.



The conjecture then predicts that there is a partition into 'isogeny classes'

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Moreover, the $S_{\phi} \subset S_{K}(G,X)(\overline{\mathbb{F}}_{p})$ have the following description ('Rapoport-Zink uniformisation')

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Here $X_p(\phi)$ is an affine Deligne-Lusztig variety of level K_p .

Previous Work

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Theorem (Kisin, 2008 and 2013)

Let (G,X) be a Shimura datum of abelian type, let p>2 and suppose that $G_{\mathbb{Q}_p}$ is unramified and that K_p is hyperspecial. Then the Langlands-Rapoport conjecture holds for (G,X,p).

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Theorem (Zhou, 2017)

Let (G,X) be a Shimura datum of Hodge type, let p>2 and suppose that $G_{\mathbb{Q}_p}$ is residually split, then isogeny classes have Rapoport-Zink uniformisation for arbitrary parahorics K_p .

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Theorem 1 (-)

Suppose that G has no factors of type A and that $\mathbf{Sh}_K(G,X)$ is proper. Then the Langlands-Rapoport conjecture holds for the Kisin-Pappas integral models of $\mathbf{Sh}_K(G,X)$.

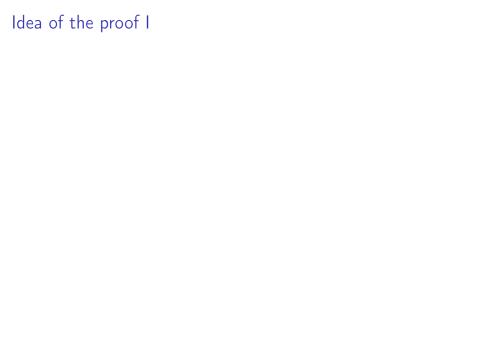
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Remarks

The assumption that $G_{\mathbb{Q}_p}$ is unramified can be removed for most (G,X).



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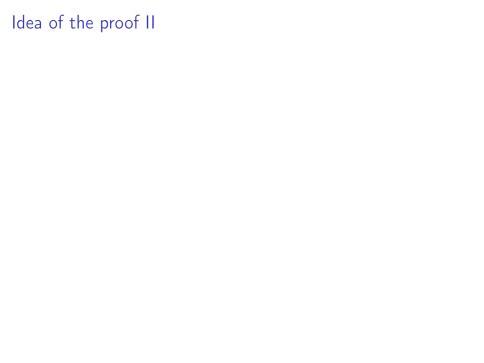
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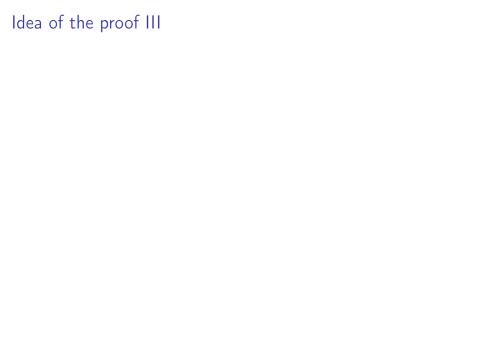
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For Hodge type Shimura varieties, the integral models do not have a moduli interpretation, which makes it difficult to make the above strategy work. We can still associate a p-divisible group with extra structures X to an $\overline{\mathbb{F}}_p$ -point, but it is no longer clear that the fiber only depends on this X.



Let K_p be a hyperspecial parahoric and $K_p' \subset K_p$ another parahoric. Let $S_{K,\overline{\mathbb{F}}_p}(G,X)$ be the special fiber of the Kisin-Pappas integral model, then it has a morphism to the 'moduli space of p-divisible groups with extra structures'.

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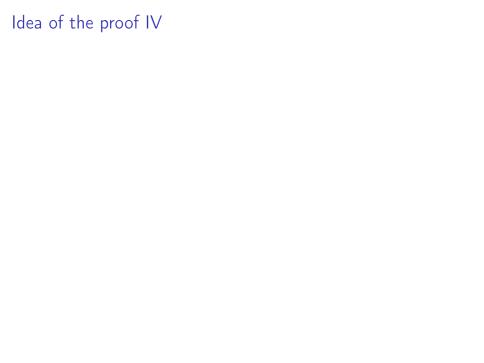
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The LR conjecture holds for the Shimura variety in the top left corner if and only if the diagram is Cartesian.



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This last result is new even for $S_{K',\overline{\mathbb{F}}_n}(G,X)!$

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