Intersection cohomology & L-functions
Cross Atlantic Representation Theory and Other topics ONline

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Abstract

I will report on ongoing joint work with Jonathan Wang, relating the intersection complex of the arc space of a spherical variety to an unramified local L-function. This is a broad generalization of Iwasawa–Tate theory ($G = \mathbb{G}_m$, $X = \mathbb{A}^1$), where the local unramified L-factors are represented by the characteristic function of the integers $\mathfrak{o}$ of a non-Archimedean field. For more general groups G and possibly singular spherical G-varieties X, the characteristic function of $X(\mathfrak{o})$ is not the correct object to consider, and has to be replaced by a function obtained as the Frobenius trace of the intersection complex of the arc space of X. In special cases of horospherical, toric, affine homogeneous spherical varieties, or certain reductive monoids, the relation of this function to L-functions was previously described in works of Braverman–Finkelberg–Gaitsgory–Mirković, Bouthier–Ngô and myself. Our current work describes these IC functions in a very general setting, relating the IC function of the arc space to an L-value determined by the geometry of the spherical variety.

1 Introduction

Let $F = \mathbb{F}_q((\varpi)) \supset \mathfrak{o} = \mathbb{F}_q[[\varpi]]$,
$X/\mathbb{F}_q$ affine, then $X(\mathfrak{o}) = \mathcal{L}X(\mathbb{F}_q)$,
$\mathcal{L}X =$ the *arc space* of X, $\mathcal{L}X(R) = \varprojlim X(R[\varpi]/\varpi^n) = \text{Maps}(D \to X)$, $(D = \operatorname{spec} \mathbb{F}_q[[\varpi]])$,
e.g., $\mathfrak{o} = \varprojlim \mathfrak{o}/\varpi^n$, and we view $\mathfrak{o}/\varpi^n$ as the $\mathbb{F}_q$-points of an n-dimensional vector space.

Goal of today's lecture:
Let $X \backsim G$ spherical, i.e., (normal &) $B \subset G$ has an open dense orbit.
Will describe a relationship between the geometry of $\mathcal{L}X$ and (unramified) L-functions for G.
Local unramified L-function: determined by a graded representation of $\check{G} = \check{G}(\mathbb{C})$, i.e., $\check{G} \times \mathbb{G}_m \xrightarrow{\rho_X} \mathrm{GL}(V_X)$ and a Satake parameter $\langle \text{Frob} \rangle \xrightarrow{\phi} \check{G}$:

$$L(\phi, \rho_X) = \prod_i \det(I - q^{-\frac{i}{2}} \rho_X^{(i)} \circ \phi(\text{Frob})).$$

The geometry will be reflected in the *intersection complex* $IC_{\mathcal{L}X}$. Through Frobenius trace (sheaf–function dictionary), it gives rise to a "basic function" in some "Schwartz space" $\Phi_0 \in \mathcal{S}(X(\mathfrak{o}))^{G(\mathfrak{o})}$.

2 Examples with smooth spaces

2.1 Iwasawa–Tate theory

$X = \mathbb{A}^1 \backsim \mathbb{G}_m$, smooth, hence

$$\Phi_0 = 1_{\mathfrak{o}} = \sum_{n \geqslant 0} 1_{\varpi^n \mathfrak{o}^\times} = \sum_{n \geqslant 0} \varpi^{-n} \cdot 1_{\mathfrak{o}^\times} \Rightarrow$$

$$\int \Phi_0(a) \chi(a) d^\times a = \sum_{n \geqslant 0} \chi(\varpi)^n = \frac{1}{1 - \chi(\varpi)} = L(\chi, 0).$$

2.2 Horospherical spaces

$X = \mathbb{A}^2 \hookleftarrow X^\bullet = N \backslash \mathrm{SL}_2$, smooth,
notice that $X(\mathfrak{o}) = \mathfrak{o}^2$, while $X^\bullet(\mathfrak{o}) = \mathfrak{o}^2 \smallsetminus \mathfrak{p}^2$,

$$\Phi_0 = 1_{X(\mathfrak{o})} = \sum_{n \geqslant 0} q^{-n} \varpi^{-n} \cdot 1_{X^\bullet(\mathfrak{o})} = \frac{1}{1 - q^{-1} \varpi^{-1} \cdot} 1_{X^\bullet(\mathfrak{o})},$$

where $a \cdot$ denotes the action of $a \in F^\times$ by scaling, normalized so that it is unitary, i.e., $a \cdot f(x,y) = |a| f(ax, ay)$.
If we integrate against an unramified character of $F^\times$, this becomes

$$\int \Phi_0 \left(\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \chi^{-1}(a) |a|^{-1} d^\times a = L(\chi, 1).$$

Better said, the torus $A = B/N$ acts on X, and $\varpi^{-1} \cdot$ is (normalized) translation by $e^{\check{\alpha}}(\varpi)$, so

$$\Phi_0 = \frac{1}{1 - q^{-1} e^{\check{\alpha}}(\varpi)} \cdot 1_{X^\bullet(\mathfrak{o})}.$$

More generally, we'll write functions on $N\backslash G(F)/K$ ($K = G(\mathfrak{o})$) as series in the cocharacter lattice (using exponential notation); a term $\frac{1}{1-q^{-s}e^{\check\lambda}}$ gives rise to $L(\chi, \check\lambda, s)$ after (normalized) integration against the unramified character χ^{-1} of $A = B/N$.

(This calculation is familiar from a global comparison of Eisenstein series: we can define

$$E(z,s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} \text{ vs. } E^*(z,s) = \sum_{(m,n)\neq(0,0)} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}$$

Then $E^*(z,s) = \zeta(2s+1)E(z,s) = L(\delta^s \circ e^{\check\alpha}, 1)E(z,s)$.)

2.3 Hecke period

$X = \mathbb{G}_m\backslash \mathrm{PGL}_2 = \begin{pmatrix} * & \\ & * \end{pmatrix} \backslash \mathrm{PGL}_2$, $\Phi_0 = 1_{X(\mathfrak{o})}$.

Various related ways to extract L-values out of this function:

- $W_X(g) := \int_F \Phi_0\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi^{-1}(x)dx \in C^\infty((N,\psi)\backslash G)^K$, and interpret the output in terms of the Casselman–Shalika formula; this is directly related to the calculation of global period integrals over the torus in terms of Fourier coefficients of modular forms:

$$\int_{k^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a = \int_{\mathbb{A}^\times} \mathrm{Whitt}_f \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a.$$

- $P_X(g) := \int_F \Phi_0\left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx \in C^\infty(N\backslash G)^K$,

  calculate: $P_X = 1_{NK} + 2\sum_{n\geqslant 1} q^{-n} 1_{N\begin{pmatrix} \varpi^n & \\ & 1\end{pmatrix}K} = (1+q^{-\frac{1}{2}}\varpi\cdot)\left(\sum_{n\geqslant 0} q^{-n} 1_{N\begin{pmatrix} \varpi^n & \\ & 1\end{pmatrix}K}\right) =$

$$= (1+q^{-\frac{1}{2}}e^{\frac{-\check\alpha}{2}}(\varpi))\sum_{n\geqslant 0} q^{-\frac{n}{2}} e^{-n\frac{\check\alpha}{2}(\varpi)} \cdot 1_{NK} = \frac{1+q^{-\frac{1}{2}}e^{\frac{-\check\alpha}{2}}(\varpi)}{1-q^{-\frac{1}{2}}e^{\frac{-\check\alpha}{2}}(\varpi)} \cdot 1_{NK} = \frac{1-q^{-1}e^{-\check\alpha}(\varpi)}{(1-q^{-\frac{1}{2}}e^{\frac{-\check\alpha}{2}}(\varpi))^2} \cdot 1_{NK}.$$

  Explanation of this calculation: $Y := X/N = \mathbb{G}_m\backslash \mathrm{PGL}_2/N = \mathbb{G}_m\backslash \mathrm{SL}_2/N = (\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{G}_m$, $\mathbb{G}_m$ acts as $(x,y)\cdot a = (ax, a^{-1}y)$. The quotient is a non-separated scheme, isomorphic to the affine line with doubled origin. At the level of $\mathfrak{o}$-points, it is the union of $\mathfrak{o}$ with $\mathfrak{o}$ over the common open subset $\mathfrak{o}^\times$. The integral is computing a pushforward $X/N \to X /\!\!/ N = \mathbb{A}^1$ of $1_{Y(\mathfrak{o})}$; the value is 1 on $\mathfrak{o}^\times$, and 2 on $\mathfrak{p}$ (up to the measure factor q^{-n}).

- Plancherel formula: The above integral $P_X = R_X(\Phi_0)$, where R_X is the "X-Radon-transform" = integral over generic horocycles (=N-orbits) on X. If we combine this with the Radon transform/standard intertwining operator R_0 for $N^-\backslash G$: $R_0(\phi)(g) = \int_N \phi(ng)dn$,

$$C_c^\infty(X) \xrightarrow{R_X} C^\infty(N\backslash G) \xleftarrow{R_0} C_c^\infty(N^-\backslash G)$$

then $B_0 := R_0^{-1} \circ R_X : C_c^\infty(X) \to C^\infty(N^-\backslash G)$ *Bernstein asymptotics* map, determines the Plancherel decomposition for $C_c^\infty(X)$; for Φ_0,

$$B_0(\Phi_0) = \frac{1 - e^{-\check{\alpha}}(\varpi)}{(1 - q^{-\frac{1}{2}} e^{\frac{-\check{\alpha}}{2}}(\varpi))^2} \cdot 1_{N^- K}$$

(because $R_0(1_{N^-K}) = \frac{1-q^{-1}e^{-\check{\alpha}}(\varpi)}{1-e^{-\check{\alpha}}(\varpi)} 1_{NK}$), and

$$|\Phi_0|^2 = \int_{\check{A}/W} \frac{|1 - e^{-\check{\alpha}}(\hat{\chi})|^2}{(1 - q^{-\frac{1}{2}} e^{\frac{-\check{\alpha}}{2}}(\hat{\chi}))^2 (1 - q^{-\frac{1}{2}} e^{\frac{\check{\alpha}}{2}}(\hat{\chi}))^2} d\hat{\chi} = \int_{\check{A}/W} L(\pi_\chi, \text{Std}, \frac{1}{2})^2 (|\Delta(\hat{\chi})|^2 d\hat{\chi}),$$

π_χ : the principal series representation induced from χ, $\hat{\chi} = \chi(\varpi) \in \hat{A}$ its Satake parameter, $|\Delta(\hat{\chi})|^2 d\hat{\chi}$: the Haar measure on conjucacy classes in the compact form of $\check{G}$.

2.4 Naive expectation

Open B-orbit on X: $X^\circ \simeq A_1 \backslash B$, assume A_1 contained in a torus (possibly trivial).

Let $A_X = A/A_1$, and integrate against N-orbits on X°, to obtain a function on $NA_1\backslash G/K = A_X(F)/A_X(\mathfrak{o})$:

$$C_c^\infty(X(F))^K \ni \Phi \mapsto R_X(\Phi)(a) = \int_N \Phi(x_0 n \cdot a) dn,$$

where $x_0 \in X^\circ$.

It can be expected that for "good" input Φ the output will be (up to some standard factors)

$$P_X \approx \prod_i \frac{1}{1 - q^{-s_i} e^{\check{\lambda}_i}(\varpi)} \cdot 1_{NA_1K}$$

with Mellin transform $\widehat{P_X}(\chi)\widehat{P_X}(\chi^{-1}) = L(\pi_\chi, \rho_X)$ for some graded representation ρ_X of $\check{G}_X$ (the "dual group" of X, with maximal torus $\check{A}_X$).

2.5 Basic family of examples

Let $X^\bullet$ be the quotient of $(\mathrm{SL}_2)^n$ by the subgroup H_n, where:

$$H_n = \left\{ \begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & x_n \\ & 1 \end{pmatrix} \middle| x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

It has an action of $G = (\mathbb{G}_m \times (\mathrm{SL}_2)^n)/\pm 1$.

- $n = 1$, <u>Hecke:</u> The "naive expectation" holds in this case for $\Phi_0 = 1_{X^\bullet(\mathfrak{o})}$, as (essentially) we saw above, with $\rho_X = \text{Std} \oplus \text{Std}^\vee \backsim \check{G} = \mathrm{GL}_2$.
  It corresponds to the global Hecke period $\int_{[\mathbb{G}_m]} f \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^s da$, represents $L(\pi, \text{Std}, \frac{1}{2} + s)$.
- $n = 2$, <u>Rankin–Selberg:</u> The "naive expectation" holds in this case for $\Phi_0 = 1_{X^\bullet(\mathfrak{o})}$ or $1_{X(\mathfrak{o})}$, where $X^\bullet \hookrightarrow X = \mathbb{A}^2 \times^{\mathrm{GL}_2} G$, with $\rho_X = \text{Std} \otimes \text{Std} \oplus \text{Std}^\vee \otimes \text{Std}^\vee$.
  It corresponds to the global Rankin–Selberg period $\int_{[\mathrm{GL}_2]} f_1(g) f_2(g) E^*(g, \frac{1}{2} + s) dg$, represents $L(\pi_1 \times \pi_2, \frac{1}{2} + s)$.
- $n = 3$, the "naive expectation" doesn't work for $1_{X^\bullet(\mathfrak{o})}$: although one would expect to get $L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2} + s)$, there is a numerator which doesn't correspond to an L-function. *However,* $X^\bullet \hookrightarrow X^\dagger = [S,S]\backslash \mathrm{Sp}_6$ (one of the "low rank accidental isomorphisms" for spherical varieties, and the expectation holds for $1_{X^\dagger(\mathfrak{o}))}$ *which however is not compactly supported on* $X(F)$.
  It corresponds to the global integral of <u>Garrett</u>: $\int_{[G]} f(g) E_{\text{Siegel}}(g, \frac{1}{2} + s) dg$, represents $L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2} + s)$.
- $n \geqslant 4$?

3 Affine embeddings and IC functions

Our naive expectation is missing some ingredients:

1. X should be *affine* (the case of $U_P\backslash G$, $[P,P]\backslash G$ is also OK, because it differs from its affine completion by an L-function, but this is the only such case);
2. If X is singular, $1_{X(\mathfrak{o})}$ should be replaced by $\Phi_0 =$ *the IC function*.

3.1 Affine embeddings

As we saw, the choice of embedding matters, e.g., $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ in Tate's thesis, the embedding is responsible for an extra factor of $L(\chi, \frac{1}{2})$. (Shifted by $\frac{1}{2}$ here, by considering L^2-normalized action.)

Direct generalizations:

- $X^\bullet = \mathrm{GL}_n \backsim \mathrm{GL}_n^2$, $X^\bullet \hookrightarrow X = \mathrm{Mat}_n$, Godement–Jacquet, the embedding is responsible for an extra factor of $L(\pi, \frac{1}{2})$.
- (Split) affine toric variety $T \hookrightarrow \bar{T}$, determined by a saturated, finitely generated, strictly convex submonoid $\mathfrak{c} \subset \check{\Lambda} = \mathrm{Hom}(\mathbb{G}_m, T)$.
  If $\mathfrak{c} = \mathbb{N}^r$, $\bar{T}$ is a product of $\mathbb{G}_m$'s and $\mathbb{G}_a$'s (the latter indexed by the basis elements $\check{\lambda}_1, \ldots, \check{\lambda}_r$ of $\mathfrak{c}$).
  Then $1_{\bar{T}(\mathfrak{o})}$ corresponds to $\prod_i L(\chi, \check{\lambda}_i, \frac{1}{2})$.
  But if $\mathfrak{c}$ *is not free* $\Leftrightarrow$ $\bar{T}$ *is singular,* $1_{\bar{T}(\mathfrak{o})}$ its Mellin transform is not an L-function.
- Back to GL_n and more general reductive groups H, there exist various $H \times H$-equivariant affine embeddings ("reductive monoids") $H \hookrightarrow \bar{H}$, e.g., the L-monoids (Ngô) determined by a heighest weight $\check{\lambda}$ for the dual group. Almost all singular.

3.2 The IC function

Now we start thinking of $X(\mathfrak{o})$ as $\mathcal{L}X(\mathbb{F}_q) = \mathrm{Maps}(D \to X)$, $D = \mathrm{spec}\,\mathbb{F}_q[[\varpi]]$. This is an *infinite-dimensional ind-scheme*, and thus does not have a good theory of perverse sheaves. (May be OK soon, based on recent work of Bouthier–Kazhdan.) However, Grinberg–Kazhdan and Drinfeld proved that in a formal neighborhood of a *non-degenerate arc* $\gamma : D \to X$ (i.e., D^* lies in the smooth locus X^{sm}), the singularities are of finite type:

$$\mathcal{L}X_\gamma \simeq Y_{\gamma'} \times D^\infty,$$

where $\gamma' \in Y$: a scheme of finite type. This allows one to define the *IC function* as

$$\Phi_0(\gamma) = \mathrm{tr}(\mathrm{Frob}^{-1}, IC^Y_{\gamma'}[-\dim Y]),$$

where IC^Y is the intersection complex of Y (a perverse sheaf obtained as the intermediate extension of the constant sheaf on the smooth locus). One can show [Bouthier–Ngô–S.] that $\Phi_0 \in C^\infty(X^{sm}(F) \cap X(\mathfrak{o}))$ is independent of the model Y chosen.

3.3 BFGM

Example: $X = \overline{N\backslash G}^{\mathrm{aff}} = \mathrm{spec}\,\mathbb{F}_q[N\backslash G]$, G simply connected, then Braverman–Finkelberg–Gaitsgory–Mirković have computed:

$$\Phi_0 = \prod_{\check{\alpha}>0} \frac{1}{1 - q^{-1}e^{\check{\alpha}}} \cdot 1_{NK} = L(\check{\mathfrak{n}}, 1),$$

i.e., supported on the negative coroot lattice, and equal to a deformation of Kostant's partition function:

$$\Phi_0(\check{\lambda}(\varpi)) = q^{-\langle \check{\lambda}, \rho \rangle} \sum_P q^{-|P|}, \text{ where } P \text{ runs over all partitions of } \check{\lambda} \text{ into a sum of negative roots.}$$

3.4 Global models

To produce the finite-dimensional models of the Grinberg–Kazhdan–Drinfeld theorem, we can replace $\mathcal{L}X = \text{Maps}(D \to X)$ with $M_X = \text{Maps}(C \to X/G)$, the stack classifying G-bundles $\mathscr{G}$ on a smooth projective curve C, together with a G-equivariant morphism $\sigma : \mathscr{G} \to X$. Fixing a point $c \in C$, we have a formally smooth cover $\hat{M}_X \to M_X$, where $\hat{M}_X$ denotes the above data together with a trivialization of $\mathscr{G}$ on the formal neighborhood D_c, and $\hat{M}_X$ maps to $\mathcal{L}X$. If, for $\gamma' = (\mathscr{G}, \sigma) \in M_X$ with $\sigma|_{C \smallsetminus \{c\}}$ in X^{sm}, then the map $\hat{M}_X \to \mathcal{L}X$ is formally smooth at every preimage of γ'.

Upshot: To compute the IC function for $\mathcal{L}X$, it suffices to compute the stalk of IC_{M_X} at such a point γ'.

3.5 Example: models for toric varieties (Bouthier–Ngô–S.)

First, consider $X = \mathbb{A}^1 \curvearrowleft G = \mathbb{G}_m$. The global model $M_X^\bullet$ (where $^\bullet$:= generically in the open G-orbit) classifies line bundles on C together with a section, hence is the scheme $\text{Sym}^\bullet C$ of effective divisors on X.

For a torus T and a smooth toric variety X described by a monoid $\mathfrak{c}_X \simeq \mathbb{N}^r \subset \text{Hom}(\mathbb{G}_m, T)$, we similarly have $M_X^\bullet = (\text{Sym}^\bullet C)^r$, the scheme of $\mathfrak{c}_X$*-valued divisors*.

If X is not smooth $\Leftrightarrow$ $\mathfrak{c}_X$ is not free, the scheme of $\mathfrak{c}_X$*-valued divisors* turns out to be singular. For every $\check{\lambda} \in \mathfrak{c}_X$ (representing an orbit in $(X(\mathfrak{o}) \cap X^\bullet(F))/G(\mathfrak{o})$), there are several irreducible components intersecting at the point $\gamma' \in M_X$ as above. They are indexed by partitions P of λ into the *indecomposable* elements $\check{\lambda}_i$ of $\mathfrak{c}_X$, and for each such partition, the normalization of the component is equal to $\text{Sym}^{|P|}C$.

It follows that the IC function is $\Phi_0(\check{\lambda}(\varpi)) = \sum_P q^{-\frac{|P|}{2}}$, i.e.,

$$\Phi_0 = \prod_{\check{\lambda}_i:\text{ indecomposable}} \frac{1}{1 - q^{-\frac{1}{2}} e^{-\check{\lambda}_i}} \cdot 1_{T(\mathfrak{o})}.$$

3.6 Zastava models

(Zastava = flag in Croatian.)

Assume (for simplicitly of exposition) that B acts freely on the open orbit X°.

The Zastava model is defined as $Z_X = \text{Maps}(C \to X/B)^\circ$ and is a *scheme* under the above hypotheses. Moreover, if we consider the toric variety $X /\!\!/ N$ for $A = B/N$, we have a natural map $Z_X \to M^\bullet_{X/\!\!/ N} = Z_{X/\!\!/ N}$ (covered in §3.5).

Example 3.6.1. For $X = \mathbb{G}_m \backslash \text{PGL}_2$, we have seen that $X/N =$ the affine line with doubled origin, so $X /\!\!/ N =$ the affine line, $Z_{X/\!\!/ N} = \text{Sym}^\bullet C$ (classifying line bundles + a non-zero section), and Z_X is its finite cover that labels each zero by one of the two points over the origin, i.e., $Z_X = \text{Sym}^\bullet C \mathring{\times} \text{Sym}^\bullet C$ (where $\mathring{\times}$ means: the divisors are disjoint).

The problem at hand, precisely formulated:

Compute the pushforward of the IC sheaf under $Z_X \to Z_{X/\!\!/ N}$.

This corresponds to the pushforward/X-Radon transform under $X(\mathfrak{o}) \to X /\!\!/ N(\mathfrak{o})$.

Remark 3.6.2. In other words, we are not asking for an explicit description of the IC function as a function on $X^\bullet(F) \cap X(\mathfrak{o})$, but for its image under the Radon transform, related to its spectral/Plancherel decomposition, that will allow us to relate it to L-functions.

4 The result and proof

From now on, we will assume that $\check{G}_X = \check{G}$. This means two things:

- B acts freely on the open orbit X°;
- for every simple root α, the PGL_2-variety $X^\circ P_\alpha/\mathcal{R}(P_\alpha)$ is isomorphic to $\mathbb{G}_m \backslash \text{PGL}_2$.

Such is, e.g., the family of examples of §2.5.

4.1 Statement

For the homogeneous part $X^\bullet = H\backslash G$, the quotient $X^\bullet /\!\!/ N$ is a toric variety for $A = B/N$, corresponding to a monoid $\mathfrak{c}_X$ of coweights; we will assume (as we may, by passing to an abelian cover) that $\mathfrak{c}_X$ is free, rank r. Its basis elements $\check\nu_i$, $i = 1, \dots, r$, correspond to the *colors*, i.e., B-stable divisors in $X^\bullet$, with $\check\nu_i$ being the valuation induced by the corresponding color on $\mathbb{F}_q(X)^{(B)}$. (Some non-degeneracy assumption here, again by passing to an abelian cover, to avoid "double points" like $\mathbb{G}_m\backslash \mathrm{PGL}_2$ — replace by $\mathbb{G}_m\backslash\mathrm{GL}_2$. Also assuming: $p \gg 1$.)

Our results distinguish between the case of the minimal affine embedding $\overline{X^\bullet}^{\mathrm{aff}} = \operatorname{spec}\mathbb{F}_q[X^\bullet]$ and other affine embeddings. I will only present the case of $X = \overline{X^\bullet}^{\mathrm{aff}}$.

We have a map $X/N \to X /\!\!/ N$. Our assumptions imply that it is an isomorphism in codimension one.

The result is best formulated for the *compactified* Zastava model: instead of

$$Z_X = \mathrm{Maps}(C \to X/B)^\circ = \mathrm{Maps}(C \to (X/N)/A)^\circ = \mathrm{Maps}(C \to (X \times N\backslash G)/A \times G)^\bullet,$$

consider

$$\bar Z_X = \mathrm{Maps}(C \to (X \times \overline{N\backslash G})/A \times G)^\bullet,$$

where $\overline{N\backslash G}$ is the affine closure of $N\backslash G$.

Let $\check\Theta$ be the set of all W-translates of the coweights $\check\nu_i$; write $\check\Theta^+$ for those that belong to the cone spanned by the $\check\nu_i$'s.

Theorem 4.1.1 (S.–Wang). *Consider the compactified Zastava model $\bar Z_X$, and its map π to the corresponding space for $X /\!\!/ N$:*

$$Z_{X/\!\!/N} = \mathit{Maps}(C \to (X /\!\!/ N)/A)^\circ = (\mathit{Sym}^\bullet C)^r \quad \text{(the space of } \mathfrak{c}_X\text{-valued divisors).}$$

If the coweights $\check\nu_i$ are minuscule, then there exists a canonical isomorphism

$$\pi_!(IC_{\bar Z_X}) \simeq \bigoplus_{\mathfrak{P}} \left(\bigotimes_{\check\Theta^+} \mathit{Sym}^{m_{\check\theta}}(\mathbb{Q}_l) \right) \otimes \iota^{\mathfrak{P}}_!(IC_{C^{\mathfrak{P}}}) \tag{4.1}$$

where $\mathfrak{P} \in \mathbb{N}^{\check\Theta^+}$, $\mathfrak{P} = (m_{\check\theta})_{\check\theta}$, $C^{\mathfrak{P}} = \prod_{\check\Theta^+} \mathit{Sym}^{m_{\check\theta}} C$, and $\iota^{\mathfrak{P}}$ is the map that sends the $\check\Theta^+$-labelled divisor $(D_{\check\theta})_{\check\theta}$ to the corresponding $\mathfrak{c}_X$-valued divisor $\sum D_{\check\theta}\check\theta$.

(Partial results for the non-minuscule case.)

4.2 Function-theoretic interpretation

Consider the map $X(\mathfrak{o}) \xrightarrow{\pi_{\mathrm{loc}}} X /\!\!/ N(\mathfrak{o})$. For every $c \in C$, an point $(\sigma, \mathscr{A}) \in Z_{X/\!\!/N} = \mathrm{Maps}(C \to (X /\!\!/ N)/A)^\circ$ gives rise to an element $\mathrm{val}(\sigma) \in (A(F) \cap X /\!\!/ N(\mathfrak{o}))/A(\mathfrak{o}) = \mathfrak{c}_X$.

The IC function Φ_0^X lives on $X(\mathfrak{o}) \cap X^{\mathrm{sm}}(F)$. We want to compute the integral over generic fibers of π_{loc}, as a function on the monoid $\mathfrak{c}_X = \bigoplus \mathbb{N}\check\nu_i$. We will do so by reading off the Frobenius trace of the sheaf $\pi_! IC_{Z_X}$ at points $(\sigma, \mathscr{A})$ with the desired valuation; if σ has non-trivial valuations $\check\lambda_j$ at various points c_j, this will give us the product $\prod_j \Phi_0^X(\check\lambda_j)$.

Write $q^{-\frac{\sum c_i}{2}} e^{\sum c_i \check\nu_i}$ for the characteristic function of $\sum c_i\check\nu_i \in \mathfrak{c}_X$. If the integral gave the constant 1 on $X /\!\!/ N(\mathfrak{o})$, we would write it as

$$\prod_i \frac{1}{1 - q^{-\frac12} e^{\check\nu_i}} = \prod_i (\sum_{n \geqslant 0} q^{-\frac{n}{2}} e^{n\check\nu_i})$$

(as in normalized Tate's thesis, up to a sign convention: $e^{\check\lambda}$ corresponds to $\check\lambda(\varpi)^{-1} \cdot 1_{A(\mathfrak{o})}$).

But this is not what we have here: we have the additional weights $\check\theta \in \check\Theta^+$, which are obtained as W-translates of the $\check\nu_i$'s. Compare with the BFGM result for $Y = \overline{N\backslash G}$ (G simply connected): while $Y /\!\!/ N$ is the toric embedding of A corresponding to *simple* positive coroots $\check\Delta \subset \check\Phi^+$, *all* positive coroots appear in the description of IC_Y:

$$\Phi_0^Y = \prod_{\check\alpha \in \check\Phi^+} \frac{1}{1 - q^{-1} e^{\check\alpha}} \cdot 1_{NK}$$

Correcting for this factor (because we used the Zastava model for $\overline{N\backslash G}$ instead of $N\backslash G$), Theorem 4.1.1 says:

Theorem 4.2.1. *The IC function for $\mathcal{L}X$ is*

$$\Phi_0^X = \frac{\prod_{\check\alpha\in\check\Phi^+}(1-q^{-1}e^{\check\alpha})}{\prod_{\check\Theta^+}(1-q^{-\frac{1}{2}}e^{\check\theta})}. \tag{4.2}$$

If we translate this to the Plancherel formula, as in §2.3, we get:

$$|\Phi_0|^2 = \int_{\check A/W} \frac{|\Delta(\hat\chi)|^2 d\hat\chi}{\prod_{\check\Theta}(1-q^{-\frac{1}{2}}e^{\check\theta})}.$$

Because of our assumption that the weights are minuscule, this reads

$$\int_{\check A/W} L(\pi_\chi, \rho_X, \frac{1}{2})(|\Delta(\hat\chi)|^2 d\hat\chi),$$

for a representation ρ_X of $\check G$ *with heighest weights translates of the colors* $\check\nu_i$.

Example 4.2.2. For the family $X^\bullet = H_n\backslash G_n$ of §2.5, if we take $X = \overline{X^\bullet}^{\text{aff}}$, we get

$$|\Phi_0|^2 = \int_{\check A/W} L(\pi_1 \times \cdots \times \pi_n, \frac{1}{2})(|\Delta(\hat\chi)|^2 d\hat\chi).$$

Remark 4.2.3. The expression of $|\Phi_0|^2$ in terms of an L-function is anticipated by a conjecture of Ben Zvi–Venkatesh–S. on the derived endomorphism ring of $IC_{\mathcal{L}X}$.

4.3 Discussion of the main theorem

(Local discussion, although in reality we are working globally.)

4.3.1 What is non-trivial about the main theorem?

1. We do not know what $IC_{\mathcal{L}X}$ is.
2. Even if we did (e.g., when X is smooth, so IC is constant), the map $X/N \to X /\!\!/ N$ is only an isomorphism in codimension 1. Over the intersections of A-divisors in $X /\!\!/ N$, this map is highly non-trivial. (Eventually, this is where the "extra" coweights $\check\theta$ in the interior of the cone spanned by the colors $\check\nu_i$ come from.)

4.3.2 How do we address these issues?

Here is where the magic of perverse sheaves comes to save us.

Theorem 4.3.3. *For the map $X \times \overline{N\backslash G} \to X /\!\!/ N$, the corresponding global map $\bar Z_X = Maps(C \to (X \times \overline{N\backslash G})/A \times G)^\bullet \xrightarrow{\pi} Z_{X/\!\!/N} = Maps(C \to (X /\!\!/ N)/A)$ is* proper *and* stratified semi-small.

As a corollary, $\pi_! IC_{\bar Z_X}$ is a direct sum of simple perverse sheaves.

This allows us to circumvent the question of an explicit description of $IC_{\bar Z_X}$. By perversity and dimension considerations, the direct summands of $\pi_! IC_{\bar Z_X}$ will actually turn out to be constant sheaves on strata of $Z_{X/\!\!/N} \simeq (\mathrm{Sym}^\bullet C)^r$, which generically represent the fundamental class of the fiber. Thus, we know the answer once we

> find the points on $Z_{X/\!\!/N}$ where the dimension or number of irreducible components of the fiber of $\bar Z_X$ jumps.

The scheme $\bar Z_X$ has a factorization property which allows us to reduce the question to "diagonals" $C \hookrightarrow \prod_{i=1}^r \mathrm{Sym}^{m_i} C$. Once we determine the "new" contributions $IC_{C^{\mathrm{diag}}}$ on those diagonals, their symmetric powers (as in Theorem 4.1.1) will be provided "for free" by factorization.

Finally, the fact that these new contributions appear exactly for $\check\theta \in \check\Theta^+$ follows from a *functional equation*:

Theorem 4.3.4. *For any simple reflection $w_\alpha \in W$, the "new" contributions of $\check\theta$ and $w_\alpha\check\theta$ are equal, unless one of them is equal to a color $\check\nu_i$.*

This is proven using the fact that $X^\circ P_\alpha/\mathcal{R}(P_\alpha) \simeq \mathbb{G}_m\backslash \mathrm{PGL}_2$, by reduction to embeddings of $\mathbb{G}_m\backslash\mathrm{PGL}_2$.