On the Stable Transfer for Sym^n Lifting of GL_2: Archimedean Case

Zhilin Luo

University of Minnesota

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- The local class field theory says that there is a canonical continuous group homomorphism

$$a_F : W_F \to F^\times$$

inducing topological isomorphism $W_F^{\mathrm{ab}} \simeq F^\times = \mathrm{GL}_1(F)$.

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- Weil-Deligne group $\mathcal{W}_F = \begin{cases} W_F, & F \text{ Archimedean} \\ W_F \times \mathrm{SL}_2(\mathbb{C}), & F \text{ } p\text{-adic} \end{cases}$

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- **Local Langlands parameters**: $\Phi_G(F)$ consists of equivalence classes of admissible homomorphisms $\varphi : \mathcal{W}_F \to {}^L G$;
- **Irreducible admissible representations**: $\mathrm{Irr}(G(F))$ (Casselman-Wallach if F is Archimedean);

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- The **Local Langlands Reciprocity** for GL is now a Theorem. When F is p-adic, it is proved by Harris-Taylor, Henniart and Scholze; When F is Archimedean, it follows from the work of Langlands on the classification of irreducible admissible representations (for general reductive Lie groups);

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- In general, the set $\Phi_G(F)$ gives a partition of $\mathrm{Irr}(G(F))$. For any $\varphi \in \Phi_G(F)$, there is a finite subset (L-**packet**) $\Pi_\varphi \subset \mathrm{Irr}(G(F))$. For GL the L-packets are singleton.

Local Langlands Functoriality

Given $\rho : {}^L H \to {}^L G$, based on the **Local Langlands Reciprocity**, there is the
Local Langlands Functoriality for ρ:
For any L-packet Π_φ of $H(F)$ associated to $\varphi \in \Phi_H(F)$, there is an L-packet $\Pi_{\rho\circ\varphi}$ of $G(F)$ that lifts Π_φ along ρ.

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- For non-isomorphic irreducible admissible representations, their distribution characters are linearly independent.

A natural question

R. Langlands proposed the following question (*Singularités et transfert*, 2013):
Given $\rho : {}^L H \to {}^L G$, does there exist a (**stable**) distribution Θ^ρ on $H(F) \times G(F)$ (more precisely their Steinberg-Hitchin bases), such that for any **tempered** L-packet Π_φ, the following identity holds

$$\mathrm{tr}_{\Pi_{\rho\circ\varphi}}(g) = \int_{H(F)} \Theta^\rho(h, g)\mathrm{tr}_{\Pi_\varphi}(h)$$

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- $\mathrm{tr}_{\Pi_\varphi} = \sum_{\pi\in\Pi_\varphi} \mathrm{tr}_\pi$, and in general it is expected that $\mathrm{tr}_{\Pi_\varphi}$ is a stable distribution on $H(F)$.

Known results

- When F is p-adic, H is an elliptic torus in $G = \mathrm{SL}_2$, and $\rho : {}^L H \to {}^L G$ is the standard embedding, the pioneer work of I. Gelfand, M. Graev and I. Pyatetski-Shapiro showed

$$\Theta^\rho(g, h) = 2\frac{\mathrm{sgn}_E(\mathrm{tr}(g) - \mathrm{tr}(h))}{|\mathrm{tr}(g) - \mathrm{tr}(h)|}$$

where E is the quadratic extension associated to H and sgn_E is the quadratic character of $F^\times$ obtained via CFT.

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- When the residue characteristic of F is not equal to 2, Langlands analyzed the question when H is a maximal torus in $G = \mathrm{SL}_2$, and $\rho : {}^L H \to {}^L G$ is the standard embedding.
- The result of G-G-PS was generalized by D. Johnstone in his PhD thesis for H an unramified elliptic torus in $G = \mathrm{SL}_\ell$, under mild assumptions on the residue characteristic of F and character formulas for the associated supercuspidal representations of $\mathrm{SL}_\ell(F)$, where ℓ is a prime.

Non-abelian situation

What about the case when both H and G are non-abelian?

Focus

$H = \mathrm{GL}_2$, $G = \mathrm{GL}_{n+1}$, $\rho = \mathrm{Sym}^n$;

- Local Langlands Reciprocity for GL is known;
- L-packets are singleton, stability for GL is equivalent to conjugation invariance;
- The classification of tempered representations for GL is known. p-adic by H. Jacquet, Archimedean by A. Knapp and G. Zuckerman.

Preparations

Notation

- $T_n \subset B_n \subset \mathrm{GL}_n$;
- $M_n \subset P_n \subset \mathrm{GL}_n$ corresponds to partition $(2, 2, ..., 2)$ if n is even, and $(2, 2, ..., 2, 1)$ is n is odd;
- $D_{\mathrm{GL}_n}(x) = |\,\mathrm{det}(1 - \mathrm{Ad}(x))|_{\mathfrak{gl}_n/(\mathfrak{gl}_n)_x}|$ the Weyl discriminant;

When $F = \mathbb{C}$

- Tempered representations of $\mathrm{GL}_2(\mathbb{C})$ are all tempered principal series, i.e. $\pi \simeq \mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\chi_1, \chi_2)$ where χ_1, χ_2 are unitary characters of $\mathbb{C}^\times$;

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- The lifting $\rho(\pi)$ of π along ρ is given by

$$\mathrm{Ind}_{B_{n+1}}^{\mathrm{GL}_{n+1}}(\chi_1^n, \chi_1^{n-1}\chi_2, ..., \chi_2^n).$$

When $F = \mathbb{C}$

By the explicit character formulas for principal series, it is not hard to derive the following result.

Proposition (Johnstone-L.)

Define Θ^{ρ} as the parabolic induction of $\Theta^{\rho,T}$ from $T_{n+1}(\mathbb{C}) \times T_2(\mathbb{C})$ to $\mathrm{GL}_{n+1}(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, where $\Theta^{\rho,T}$ is supported on $T^{\mathrm{rss}}_{n+1}(\mathbb{C}) \times T^{\mathrm{rss}}_2(\mathbb{C})$ given by

$$\Theta^{\rho,T}(D,t) = \delta_{\prod_{k=1}^{n+1} D_k^{n+1-k}}(t_1) \otimes \delta_{\prod_{k=1}^{n+1} D_k^{k-1}}(t_2)$$

with $(D,t) = (D_1, ..., D_{n+1}, t_1, t_2)$. Then

$$D^{\frac{1}{2}}_{\mathrm{GL}_{n+1}}(g)\mathrm{tr}_{\rho(\pi)}(g) = \int_{\mathrm{GL}_2(F)} \Theta^{\rho}(g,\gamma) D^{\frac{1}{2}}_{\mathrm{GL}_2}(\gamma)\mathrm{tr}_{\pi}(\gamma)$$

for any tempered representation π of $\mathrm{GL}_2(\mathbb{C})$.

When $F = \mathbb{R}$

Tempered representations

- Tempered principal series $\mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\chi_1, \chi_2)$ where χ_1, χ_2 are unitary characters of $\mathbb{R}^\times$;
- Discrete series $D_{l,t}$ parametrized by $(l, t) \in \mathbb{N} \times i\mathbb{R}$.

Issue

The distribution constructed before does **NOT** work for discrete series.

Question

How to unify the tempered principal series and discrete series?

When $F = \mathbb{R}$

Observation

Let ω_π be the central character of π. The lifting $\rho(\pi)$ for any tempered representation π is always of the form $\mathrm{Ind}_{P_{n+1}}^{\mathrm{GL}_{n+1}}(\pi_{M_{n+1}})$ where

$$\pi_{M_{n+1}} = \begin{cases} \bigotimes_{k=1}^{\frac{n}{2}} \pi_{2k} \bigotimes (\omega_\pi)^{\frac{n}{2}}, & \text{if } n \text{ is even}, \\ \bigotimes_{k=1}^{\frac{n+1}{2}} \pi_{2k-1}, & \text{if } n \text{ is odd}. \end{cases}$$

if $\pi = \mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\chi_1, \chi_2)$. Here $\pi_m = (\omega_\pi)^{\frac{n-m}{2}} \otimes \mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\chi_1^m, \chi_2^m)$;

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if $\pi = D_{l,t}$. Here $\pi_m = (\omega_\pi)^{\frac{n-m}{2}} \otimes D_{mk,mt}$.

When $F = \mathbb{R}$

Reduce to M_{n+1}

By the character formula for induced representations, we only need to determine a distribution $\Theta^{M_{n+1}}$ on $M_{n+1}(F) \times \mathrm{GL}_2(F)$ which yields the lifting of distribution characters from π to $\pi_{M_{n+1}}$.

When $F = \mathbb{R}$: Character relation

Moreover, we have the following theorem relating character distributions between π and π_m.

Theorem (Johnstone-L.)

- *Fix a tempered principal series* $\mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\chi_1, \chi_2)$ *of* $\mathrm{GL}_2(F)$. *Then the following equality of locally integrable distributions holds*

$$D_{\mathrm{GL}_2}^{\frac{1}{2}}(\gamma^k)\mathrm{tr}_{\mathrm{Ind}(\chi_1,\chi_2)}(\gamma^k) = D_{\mathrm{GL}_2}^{\frac{1}{2}}(\gamma)\mathrm{tr}_{\mathrm{Ind}(\chi_1^k,\chi_2^k)}(\gamma)$$

  for any $k \in \mathbb{N}$.

- *When* $F = \mathbb{R}$, *let* $D_{l,t}$ *be the discrete series representation of* $\mathrm{GL}_2(\mathbb{R})$ *associated to* $(l, t) \in \mathbb{N} \times \mathbb{C}$. *Then the following equality of locally integrable distributions holds*

$$D_{\mathrm{GL}_2}^{\frac{1}{2}}(\gamma^k)\mathrm{tr}_{D_{l,t}}(\gamma^k) = D_{\mathrm{GL}_2}^{\frac{1}{2}}(\gamma)\mathrm{tr}_{D_{kl,kt}}(\gamma)$$

  for any $k \in \mathbb{N}$ **odd**.

When $F = \mathbb{R}$: Reduction formula

Assumption

When n is **odd**, assume that the distribution $\Theta^{\Delta_{\frac{n+1}{2}}}$ exists where $\Delta_{\frac{n+1}{2}}$ is the diagonal embedding from $\mathrm{GL}_2(\mathbb{C}) \times W_F$ to ${}^L M_{n+1} \simeq (\prod_{i=1}^{\frac{n+1}{2}} \mathrm{GL}_2(\mathbb{C})) \times W_F$,

Definition

Define the distribution $\Theta^{M_{n+1}}$ on $M_{n+1}(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \simeq (\prod_{i=1}^{\frac{n+1}{2}} \mathrm{GL}_2(\mathbb{R})) \times \mathrm{GL}_2(\mathbb{R})$ as follows,

$$
\begin{aligned}
&\Theta^{M_{n+1}}(g_1, \ldots, g_{\frac{n+1}{2}}, \gamma) \\
&= \Theta^{\Delta_{\frac{n+1}{2}}}\big(\det(g_1)^{\frac{n-1}{2}} \cdot g_1, \\
&\qquad \det(g_2)^{\frac{n-3}{2}} \cdot g_2^3, \ldots, \\
&\qquad \det(g_{\frac{n+1}{2}}) \cdot g_{\frac{n-1}{2}}^{n-2}, \\
&\qquad g_{\frac{n+1}{2}}^{n}, \\
&\qquad \gamma\big).
\end{aligned}
$$

When $F = \mathbb{R}$: Reduction formula

Corollary (Johnstone-L.)

The following distributional identity holds,

$$D^{\frac{1}{2}}_{M_{n+1}}(m)\mathrm{tr}_{\pi_{M_{n+1}}}(m) = \int_{\mathrm{GL}_2(F)} \Theta^{M_{n+1}}(m,\gamma) D^{\frac{1}{2}}_{\mathrm{GL}_2}(\gamma)\mathrm{tr}_{\pi}(\gamma)$$

for any tempered representation π of $\mathrm{GL}_2(F)$ when F is Archimedean.

Remark

- Define Θ^{ρ} as the parabolic induction of $\Theta^{M_{n+1}}$ from $M_{n+1}(F) \times \mathrm{GL}_2(F)$ to $\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_2(F)$.

When $F = \mathbb{R}$: Reduction formula

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Remark

- Define Θ^ρ as the parabolic induction of $\Theta^{M_{n+1}}$ from $M_{n+1}(F) \times \mathrm{GL}_2(F)$ to $\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_2(F)$.
- The functoriality for $\Delta_{\frac{n+1}{2}}$ is given by $\pi \to \bigotimes_{i=1}^{\frac{n+1}{2}} \pi$, which is much easier than ρ. However, the existence of $\Theta^{\Delta_{\frac{n+1}{2}}}$ seems to be not easy from analytical point of view.

When F is p-adic

We focus on the case when the residue characteristic of F is **NOT** equal to 2.

Tempered representations

- Tempered principal series;
- Twisted Steinberg representations with unitary central character;
- Supercuspidal representations with unitary central character;

Focus

Supercuspidal representations.

When F is p-adic: Supercuspidal representations

- By R. Howe, the supercuspidal representations of $\mathrm{GL}_2(F)$ can be parametrized by admissible pairs $(E/F, \theta)$ with E/F a tamely ramified quadratic extension, and θ an admissible character of $E^\times$. For each admissible pair $(E/F, \theta)$, we denote the corresponding supercuspidal representation by π_θ;

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- The functorial lifting $\rho(\pi)$ of $\pi = \pi_\theta$ is also of the form $\mathrm{Ind}_{P_{n+1}}^{\mathrm{GL}_{n+1}} \pi_{M_{n+1}}$, where

$$\pi_{M_{n+1}} = \begin{cases} \bigotimes_{k=1}^{\frac{n}{2}} \pi_{2k} \bigotimes (\theta|_{F^\times})^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ \bigotimes_{k=1}^{\frac{n+1}{2}} \pi_{2k-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Here $\pi_m = (\theta|_{F^\times})^{\frac{n-m}{2}} \otimes \pi_{\theta^m}$ where π_{θ^m} is the tempered representation of $\mathrm{GL}_2(F)$ with tempered L-parameter $\mathrm{Ind}_{W_E}^{W_F} \theta^m$. Note: $\theta|_{F^\times}$ is the central character of π_θ.

When F is p-adic: Character relation

Similar to the Archimedean case, we have the following relation between character distributions of π_{θ^k} and π_θ.

- Let π_θ be the supercuspidal representation of $\mathrm{GL}_2(F)$ associated to an admissible pair $(E/F, \theta)$. Then the following equality holds as locally integrable distributions, whenever $k \in \mathbb{N}$ is coprime to $(q-1)q(q+1)$, where q is the cardinality of the residue field of F

$$D^{\frac{1}{2}}_{\mathrm{GL}_2}(\gamma^k)\mathrm{tr}_{\pi_\theta}(\gamma^k) = D^{\frac{1}{2}}_{\mathrm{GL}_2}(\gamma)\mathrm{tr}_{\pi_{\theta^k}}(\gamma).$$

Remark

Under mild assumptions on k and q, except twisted Steinberg representations, the distribution characters of tempered representations behave like characters (for abelian groups).

Thank you!