A twisted Yu construction and Harish-Chandra characters

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A tale of three sign characters

$\epsilon_\sharp$ $\epsilon_\flat$ ϵ_f

$$G \supset T \xrightarrow{\epsilon_*} \{\pm 1\}$$

- $\epsilon_\sharp$ from Bruhat-Tits theory
- $\epsilon_\flat$ from Galois theory
- ϵ_f from Lie theory

Yu's construction of supercuspudal representations

Yu's construction

$$\left\{ \begin{array}{c} (G^0 \subset G^1 \subset \cdots \subset G^d = G) \\ \pi_{-1} \\ (\phi_0, \phi_1, \ldots, \phi_d) \end{array} \right\} \xrightarrow{\text{J.K.Yu}} \begin{array}{c} \left\{ \text{irred. s.c reps of } G \right\} \\ \pi_i = \text{c-Ind}_{K^i}^{G^i} \rho_i \end{array}$$

Yu's construction is exhaustive when

- Kim 2007: $p >> 0$ and $\text{char}(F) = 0$.
- Fintzen 2018: $p \nmid \#W$, any $\text{char}(F)$.

Harish-Chandra character formula (Adler-Spice 2008)

$$\Theta_{\pi_d}(\gamma_{<r} \cdot \gamma_{\geq r}) = \sum_{\substack{g \in G^{d-1} \backslash G / C_G(\gamma_{<r}) \\ {}^g\gamma_{<r} \in G^{d-1}}} (\text{roots of unity}) \cdot \Theta_{\pi_{d-1}}({}^g\gamma_{<r}) \widehat{\mu}(\log({}^g\gamma_{\geq r}))$$

Assumption: G^{d-1}/Z_G compact!

The toral case

Toral representations

$$(T = G^0 \subset G^1 = G, \phi_0) \mapsto \pi \quad \text{toral}$$

T unramified

- Reeder 2008: Organized into L-packets
- DeBacker-Spice 2018: Obstruction to stability: $\epsilon_\sharp : T(F) \to \{\pm 1\}$.

T ramified, ϕ_0 minimal depth (epipelagic)

- K. 2015: Organized into L-packets (also cf. Reeder-Yu)
- K. 2015: Obstruction to stability: $\epsilon_f : T(F) \to \{\pm 1\}$.

T general, ϕ_0 general

- K. 2019: Organized into L-packets
- K. 2019: Obstruction to stability: $\epsilon_\sharp \cdot \epsilon_f \quad \rightsquigarrow \quad$ absorb into ϕ_0

The regular case

Regular representations

Arbitrary Yu-datum, but π_{-1} regular Deligne-Lusztig $\rightsquigarrow$ π regular

K. 2019

- $\pi_{-1} \quad \leftrightarrow \quad (T, \phi_{-1}), \quad T \subset G^0$, $\phi_{-1} : T \to \mathbb{C}^\times$. Set $\theta = \prod_{i=-1}^{d} \phi_i$.
- Reg. s.c. $\pi \quad \leftrightarrow \quad$ G-conj classes of (T, θ). Just like d.s. over $\mathbb{R}$.
- If $\gamma \in T$ is shallow, AS formula holds without G^{d-1}/Z_G compact.
- $\Theta_\pi(\gamma) = e(G)\epsilon(X^*(T_0)_\mathbb{C} - X^*(T)_\mathbb{C}) \sum_w \Delta_{II}^{\text{abs}}(\gamma^w)\epsilon_f(\gamma^w)\epsilon_\sharp(\gamma^w)\theta(\gamma^w)$.
- Same formula as for discrete series over $\mathbb{R}$, up to $\epsilon_f \epsilon_\sharp$.
- Sweep under the rug: Absorb $\epsilon_f \epsilon_\sharp$ into θ.
- Obtain LLC by mimicking Langlands argument over $\mathbb{R}$.
- $\varphi : W_F \to {}^L G$ strongly regular: $\text{Cent}(\varphi(I_F), \widehat{G})$ abelian.
- Functoriality for ${}^L T \xrightarrow{\chi} {}^L G$:
  $e(G)\epsilon(X^*(T_0)_\mathbb{C} - X^*(T)_\mathbb{C}) \sum_w \Delta_{II}^{\text{abs}}[\chi](\gamma^w)\theta_\chi(\gamma^w)$.

The rug strikes back

Spice: Drop compactness from AS formula for arbitrary γ

- Mistake in Yu's paper! Due to typo in Gerardin 1977.
- Yu's intertwining proof fails. Irreducibility proof fails.
- Fintzen 2018: Intertwining claim false, but irreducibility still true.
- But: still need intertwining for character formula. Obstruction: $\epsilon_\sharp$.
- Hope: $\epsilon_\sharp$ extends from T to G_x^0.

K: Construct LLC for *arbitrary* supercuspidal parameters, $p \nmid \#W$

- π_{-1} no longer regular, must pass from G^0 to G rather than T to G.
- Dually, must replace ${}^LT \to {}^LG$ with ${}^LG^0 \to {}^LG$. Introduces: $\epsilon_\flat$.
- Hope: $\epsilon_\flat$ extends from T to G_x^0.

Dashed hopes

Neither of $\epsilon_\sharp$, $\epsilon_\flat$ extends from T to G_x^0.

Formulas

$$\epsilon_\sharp(\gamma) = \prod_{\substack{\alpha\in R(T,G)_{\mathrm{asym}}/\Sigma \\ \alpha(\gamma)\neq 1 \\ r/2\in\mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{k_\alpha^\times}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha\in R(T,G)_{\mathrm{sym,unram}}/\Gamma \\ \alpha(\gamma)\neq 1 \\ r/2\in\mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{k_\alpha^1}(\alpha(\gamma)).$$

$$\epsilon_\flat(\gamma) = \prod_{\substack{\alpha\in R(T,G)_{\mathrm{asym}}/\Sigma \\ \alpha(\gamma)\neq 1 \\ \alpha_0\in R(Z^0,G)_{\mathrm{sym,ram}} \\ 2\nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha^\times}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha\in R(T,G)_{\mathrm{sym,unram}}/\Gamma \\ \alpha(\gamma)\neq 1 \\ \alpha_0\in R(Z^0,G)_{\mathrm{sym,ram}} \\ 2\nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha^1}(\alpha(\gamma)).$$

Main technical result

Theorem (FKS 2019)

There exists a canonical sign character $\epsilon : G_x^0 \to \{\pm 1\}$ *such that for every tame maximal torus* $T \subset G$ *with* $x \in \mathcal{B}(T)$ *one has*

$$\epsilon\big|_T = \epsilon_\sharp \cdot \epsilon_\flat \cdot \epsilon_f \cdot \epsilon_{aux}.$$

Remarks

- ϵ_{aux} is very benign, easily extends to G_x^0.
- ϵ_f came for free.

Construction of ϵ

Piece #1

H aff. alg. group over k, possibly disconnected, H° reductive

β H-invariant non-degenerate symmetric bilinear form on $\mathfrak{h} = \mathrm{Lie}(H)$

$\mathrm{Ad} : H(k) \to O(\mathfrak{h}, \beta)(k) \xrightarrow{sp} k^\times/k^{\times,2} \dashrightarrow \{\pm 1\}$

Piece #2

H aff. alg. group over k, possibly disconnected

$X^*(H) \xrightarrow{2} X^*(H) = \mathrm{Hom}_{\bar{k}}(H, \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m)$ complex

$H^1(\Gamma, \mathrm{Hom}(H, \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m)) \to \mathrm{Hom}(H(k), H^1(\Gamma, \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m)) = \mathrm{Hom}(H(k), H^1(\Gamma, \mu_2)) = \mathrm{Hom}(H(k), k^\times/k^{\times,2})$

Piece #3

$$\mathrm{sgn}_{k^\times}\left(\det\left(-\mid \bigoplus_{\alpha_0 \in R(Z^0,G)_{\mathrm{sym.ram}}/\Gamma} \bigoplus_{t \in (0, e_{\alpha_0}^{-1})} \mathfrak{g}_{\alpha_0}(F^u)_{x,t:t+}\right)\right)$$

The rewards

A twisted Yu construction

Replace ρ_i by $\rho_i \otimes \epsilon$ in Yu's construction.
Reward: Yu's intertwining claims now hold.

Theorem (Spice, in preparation)

Assume π_{-1} is a Deligne-Lusztig induction from $T \subset G^0$.

$$\Theta_\pi(\gamma_0 \cdot \gamma_{>0}) = \sum_{\substack{g \in T\backslash G/C_G(\gamma_0) \\ {}^g\gamma_0 \in T}} (\textit{roots of unity}) \cdot \epsilon({}^g\gamma_0)\theta({}^g\gamma_0)\widehat{\mu}(\log({}^g\gamma_{>0}))$$

Roots of unity

$\epsilon = \epsilon_\sharp \cdot \epsilon_\flat \cdot \epsilon_f \cdot \epsilon_{\rm aux}$. Recall $\epsilon_\sharp, \epsilon_\flat, \epsilon_f$ were obstructions. They now cancel.
What about $\epsilon_{\rm aux}$? $\Delta_{II}^{\rm abs}[\chi_T](\gamma) \cdot \epsilon_{\rm aux}(\gamma) = \Delta_{II}^{\rm abs}[\chi_{Z^0}](\gamma)$.

Theorem (FKS 2020)

$\Theta_\pi(\gamma_0 \cdot \gamma_{>0}) = e(G)\epsilon(X^*(T_0)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}) \sum_g \Delta_{II}^{abs}(\gamma_0^g)\theta(\gamma_0^g)\widehat{\mu}(\log(\gamma_{>0}^g))$.

The rewards, continued

LLC from G^0 to G, assuming $p \nmid \# W$

- If we understand LLC for depth-zero supercuspidal parameters, we get LLC for all supercuspidal parameters, via ${}^LG^0 \to {}^LG$.
- But we *do* understand LLC for d.z. s.c. parameters!
- Thus: We have explicit LLC for all supercuspidal parameters, including internal structure of L-packets.

Character identities

$S_\varphi = \mathrm{Cent}(\varphi(W_F), \widehat{G})$. $\mathrm{Irr}(S_\varphi) \leftrightarrow \Pi_\varphi$.

$s \in S_\varphi$: $\Theta_\varphi^s = e(G) \sum_\rho \mathrm{tr}\rho(s)\Theta_{\pi_\rho}$. $\widehat{H} = \mathrm{Cent}(s, \widehat{G})$. $\boxed{\Theta_\varphi^s(f) = \Theta_{\varphi^H}^s(f^H)}$.

Theorem (FKS 2020, $p >> 0$)

If φ is regular, character identities hold.
For general φ, $1 \to \widehat{T}^\Gamma \to S_\varphi \to \Omega \to 1$, and char id hold for $s \in \widehat{T}^\Gamma$.

Thank You!