Statistics of automorphic representations through simplified trace formulas

Rahul Dalal

UC Berkeley

May 28, 2020

Note on technical details

This subject has a lot of details that are distractions at the level of a 20 minute talk. Therefore,

- Feel free to ignore anything in gray if you aren't familiar with the subject.
- Anything in orange will be explained only intuitively and imprecisely.

Unmotivated Defintion

Definition

Let G be a reductive group over a number field F. A discrete automorphic representation for G is an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$.

- Reductive group: matrix group with nice representation theory (root and weight theory works).
  - ex. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{U}_n, \mathrm{SO}_n, \mathrm{Sp}_n$.
  - Non ex. Upper triangular matrices.

- $G(\mathbb{A}_F), G(F)$: what ring entries of matrices are in
- L^2: square-integrable functions as a unitary representation of $G(\mathbb{A}_F)$ under right-translation.

- subrepresentation: analysis issue—infinite-dimensional representations can be direct integrals instead of direct sums
- discrete: There is a definition for non-discrete

Motivation

Why do we care about such bizarre objects?

- They have a lot of handles to grab onto when studying
  - representations of algebraic groups
  - Fourier analysis
- They mysteriously encode information about so much else:
  - Number Theory: Galois representations (Langlands conjectures)
  - Computer Science: expander graphs/higher-dimensional expanders
  - Differential Geometry: spectra of Laplacians on locally symmetric spaces
  - Combinatorics: identities for the partition function
  - Finite Groups: representation theory of large sporadic simple groups (moonshine)
  - Mathematical Physics: representations of infinite-dimensional Lie algebras.

Example

If $G = \mathrm{GL}_2/\mathbb{Q}$

$$\{\text{aut. reps. for } G\} \approx \{\text{new, eigen modular/Maass forms}\}$$

- This is NOT obvious
- Key step: If K^∞ is a maximal compact subgroup at the finite places,

$$\mathrm{GL}_2(\mathbb{Q})\mathbb{R}^\times\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{SO}_2(\mathbb{R})K^\infty = \Gamma_{K^\infty}\backslash\mathcal{H}$$

where Γ_{K^∞} is some arithmetic subgroup of $\mathrm{SL}_2\mathbb{R}$ and $\mathcal{H}$ is upper-half plane

Flath Decomposition

Theorem

Let π be an automorphic representation for group G/F. Then π factors over places v of F:

$$\pi = \widehat{\bigotimes}' \pi_v$$

where each π_v is an admissible, unitary representation of $G(F_v)$.

For $G = \mathrm{GL}_2/\mathbb{Q}$:

- π_∞ is the qualitative "type" of π: modular vs. Maass, weight
- π_p relates to the p^nth Fourier coefficients of π.

Key Question: Which combinations of π_v actually appear in L^2?

Motivation

First trick to try for decomposing a representation: look at traces.

- Assume for a moment

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi) = \bigoplus_{\pi \text{ d.a.}} \pi$$

- Then if R is an operator on L^2

$$\mathrm{tr}_{L^2} R = \sum_{\pi \text{ d.a.}} \mathrm{tr}_\pi R$$

- Choose R cleverly: put restrictions on π_v at some v and probe a test function of π_w at other w $\implies$ information towards key question: distribution of π_w "in families"

Fantasy

How do we compute these traces?

- Convolution operators: f compactly supported smooth on $G(\mathbb{A}_F)$:

$$R_f : v \mapsto \int_{G(\mathbb{A}_F)} f(g) gv \, dg$$

- If $G(F)\backslash G(\mathbb{A}_F)$ is compact,

$$\operatorname{tr}_{L^2} R_f = \sum_{[\gamma]\in[G(F)]} \mathrm{Vol}(G(F)_\gamma \backslash G(\mathbb{A}_F)_\gamma) \int_{G(\mathbb{A}_F)_\gamma \backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g) \, dg$$

- conjugacy classes, volume term, orbital integral $O_\gamma(f)$

Reality

Without compactness, nothing converges. There are various truncations to use instead. Each complicates the spectral expansion and the three pieces of the geometric expansion.

coarse expansion	explicit
	elementary
fine expansion	bad abstract props.
invariant	
	inexplicit
stable	complicated
	good abstract props.
beyond endoscopic??	simple in nice cases

Discrete Series

- Restrict attention to the nicest "qualitative type" of automorphic representations$\leftrightarrow$the nicest real representations
- Discrete series: appear discretely in $L^2(G(F_\infty))$.
- Classified into L-packets Π_λ
- $G = \mathrm{GL}_2/\mathbb{Q}$
  - L-packets singletons parameterized by $k \geq 2$.
  - $\pi_\infty \in \Pi_k$ means π a holomorphic modular form of weight k.
- The invariant trace formula dramatically simplifies when restricted to representations with discrete series at infinity.

"Simple" trace formula

Theorem ([Art89])

Let G/F be a cuspidal reductive group and let Π_λ be a regular discrete series L-packet. Let $\mathcal{A}_\lambda$ be the set of automorphic representations π of G with $\pi_\infty \in \Pi_\lambda$. Then for any compactly supported smooth test function f on $G(\mathbb{A}^\infty)$

$$\sum_{\pi \in \mathcal{A}_\lambda} \mathrm{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{ell}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- "Conjugacy classes" counted with principle of inclusion-exclusion
- "Volume term"
- "Orbital integral" factored into infinite and finite places

Computing terms

Even though we are working with general reductive groups, the terms can be somewhat explicitly computed!

$$\sum_{\pi \in \mathcal{A}_\lambda} \mathrm{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{\mathrm{ell}}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- Sums: reductive group theory [ST16, §8]
- Φ: Weyl character formula + more root combinatorics [Art89]
- O_γ: Counting points moved some amount by automorphisms of Bruhat-Tits buildings [ST16, §7]/upcoming work
- f_M: The p-adic integrals are easier [ST16, §7] OR branching laws + Kato-Lusztig formula [Dal19, §5]
- a_γ: L-functions of Gross motives for red. groups [ST16, §6]

Results

This allows us to get good enough error bounds on statistics over these families for applications:

- Shin-Templier '16
  - Plancherel equidistribution w/ error bounds—equidistribution of π_v for fixed v.
  - Automorphic Sato-Tate—equidstribution of unramified π_v over all v.
  - Distributions of low-lying zeros of L-functions in families
- D. '19
  - Break up L-packet at infinity using stable trace formula and get same error bound as Shin-Templier
- Future better understanding??
  - Sarnak-Xue type bounds—density of non-tempered components at a finite place??
  - other??

Some Further Reading

- James Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290. MR 1001841
- Rahul Dalal, *Sato-tate equidistribution for families of automorphic representations through the stable trace formula*, 2019.
- Sug Woo Shin and Nicolas Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, Invent. Math. **203** (2016), no. 1, 1–177, Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok. MR 3437869

Contact info: dalal@berkeley.edu