Vanishing theorems for Shimura varieties

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2. Conjectures about vanishing of cohomology

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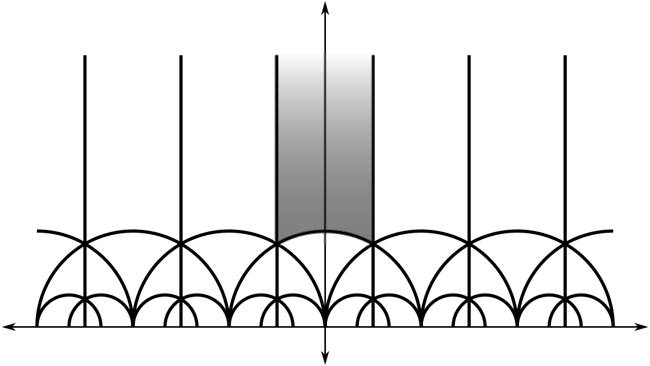

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- Beyond abelian type. Examples: those associated to Dynkin diagrams of types E_6, E_7.

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What is it good for?

- Taylor–Wiles patching.
- For modular curves, used to establish the compatibility with p-adic local Langlands ($\ell = p$).
- Useful in studying Bloch–Kato conjecture.

Let $N_0 \subset G(\mathbb{Q}_p)$ be a compact subgroup. Define

$$\widetilde{H}^i_{(c)}(K^p N_0, \mathbb{F}_p) := \varinjlim_{N_0 \subseteq K_p} H^i_{(c)}(X_{K^p K_p}(\mathbb{C}), \mathbb{F}_p).$$

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For $N_0 = \{1\}$, we also expect vanishing of $\widetilde{H}^i$ whenever $i > d$. Calegari–Emerton conjecture, motivated by heuristics from p-adic Langlands programme.

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Theorem 1 (Scholze, C-Scholze)

There exists a perfectoid space $\mathcal{X}_{K^p}^* = \varprojlim_{K_p} \mathcal{X}_{K^pK_p}^*$ and a morphism of adic spaces

$$\pi_{\mathrm{HT}} : \mathcal{X}_{K^p}^* \to \mathscr{F}\ell$$

- π_{HT} measures the relative position of the Hodge–Tate filtration. For the modular curve, we have:

  $$x \in \mathcal{X}_{K^p}(C, \mathcal{O}_C) \leftrightarrow (E, \alpha : T_pE \simeq \mathbb{Z}_p^2) \mapsto \mathrm{Lie}E(1) \subset T_pE \otimes_{\mathbb{Z}_p} C \overset{\alpha}{\simeq} C^2.$$

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- π_{HT} is "affinoid": there exists an open cover of $\mathscr{F}\ell$ by affinoid U_i such that each $\pi_{\mathrm{HT}}^{-1}(U_i)$ is affinoid perfectoid.

If (G, X) is a Shimura datum of abelian type, then

$$\widetilde{H}^i_{(c)}(K^p, \mathbb{F}_p) = 0 \text{ whenever } i > d.$$

This is a theorem of Scholze and Hansen–Johansson. The case of $\widetilde{H}^i_c(K^p, \mathbb{F}_p)$ is based on purely geometric techniques.

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Theorem 2 (C–Gulotta–Johansson + Hsu–Mocz–Reinecke–Shih)

Assume that (G, X) is a Shimura datum of Hodge type, and $G_{\mathbb{Q}_p}$ is split. Let $N_0 := N(\mathbb{Z}_p)$. Then

$$\widetilde{H}^i_c(K^p N_0, \mathbb{F}_p) = 0 \text{ whenever } i > d.$$

- p-adic Hodge theory:

$$R\Gamma_{\mathrm{et},c}\left(\mathcal{X}_{K^pN_0}, \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \overset{a}{\simeq} R\Gamma_{\mathrm{et}}\left(\mathcal{X}^*_{K^pN_0}, \mathcal{I}^+/p\right),$$

where $\mathcal{I}^+ \subseteq \mathcal{O}^+$ is the ideal of sections that vanish at the boundary.

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- The Bruhat stratification into $B(\mathbb{Q}_p)$-orbits

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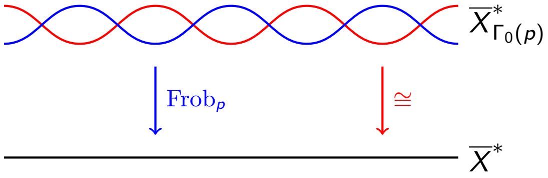
- By quantifying when different subsets of $(\mathcal{X}^*_K)_K$ become perfectoid, we show that the cohomological amplitude of $R\pi_{\mathrm{HT}/N_0,*}\mathcal{I}^{+a}/p$ restricted to $\mathscr{F}\ell^w/N_0$ lies in $[0, d - \dim \mathscr{F}\ell^w]$.

For the modular curve, the geometry of the reduction mod p:



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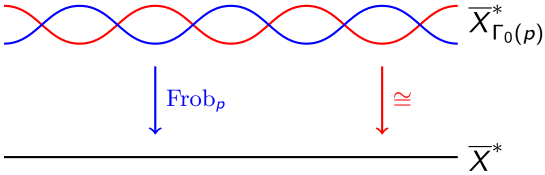

$$\overline{X}^{*,\mathrm{ord}}_{\Gamma_0(p)} = \overline{X}^{*,\mathrm{anti}}_{\Gamma_0(p)} \sqcup \overline{X}^{*,\mathrm{can}}_{\Gamma_0(p)}$$

matches the Bruhat stratification:

$$\mathbb{P}^{1,\mathrm{ad}} = \mathbb{A}^{1,\mathrm{ad}} \sqcup \{\infty\}.$$

In particular, $\mathcal{X}^{*,\mathrm{anti}}_{K^pN_0}$ is perfectoid (Ludwig).

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Theorem 3 (C-Scholze)

1. *If X_K is compact, then $H^i_{(c)}(X_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}} = 0$ unless $i = d$.*
2. *If G is quasi-split and $\mathrm{length}(\overline{\rho}_{\mathfrak{m}}) \leq 2$, then*

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- The Newton stratification

$$\mathscr{F}\ell = \sqcup_{b \in B(G,\mu)} \mathscr{F}\ell^b$$

and the identification of the fibers over $\mathscr{F}\ell^b$ with perfectoid Igusa varieties Ig^b (Mantovan product formula).

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Remark

Boyer proves a stronger result for Shimura varieties of *Harris–Taylor type*, going beyond the generic case. There is also forthcoming work of Koshikawa.

For the modular curve, we have $B(G, \mu) = \{\mathrm{ord}, \mathrm{ss}\}$:

$$\begin{array}{ccccc} \mathcal{X}^*_{K^p} & = & \mathcal{X}^{*,\mathrm{ord}}_{K^p} & \sqcup & \mathcal{X}^{\mathrm{ss}}_{K^p} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^{1,\mathrm{ad}} & = & \mathbb{P}^{1,\mathrm{ad}}(\mathbb{Q}_p) & \sqcup & \Omega \end{array}$$

- Let $D/\mathbb{Q}$ be the quaternion algebra ramified at p, ∞. The fibers $\mathrm{Ig}^{\mathrm{ss}}$ of π_{HT} over Ω can be identified with Shimura sets for $D^\times$ (Howe).

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- If π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that π_f^p contributes to $R\Gamma(\mathrm{Ig}^{\mathrm{ss}}, \mathbb{Q}_\ell)$, then π_p cannot be a principal series representation.

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Theorem 4 (C-Tamiozzo, in progress)

Let $\ell \geq 3$ *and* $\mathfrak{m} \subset \mathbb{T}$ *be in the support of* $H^*_{(c)}(X_K, \mathbb{F}_\ell)$ *such that* $\mathrm{Im}(\overline{\rho}_\mathfrak{m}) \supset \mathrm{SL}_2(\mathbb{F}_\ell)$. *Then*

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- Key idea: work with an auxillary prime p that splits completely in F. Replace the direct computation of Igusa cohomology with the geometric Jacquet–Langlands functoriality established by Tian–Xiao.