SECOND PRO-p-IWAHORI COHOMOLOGY FOR SL_2

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0.1. **Notation.** Let F denote a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, uniformizer ϖ and residue field k_F of size $q = p^f$. We suppose throughout that $p > 2e(F/\mathbb{Q}_p) + 1$. We fix an embedding $k_F \hookrightarrow \overline{\mathbb{F}}_p$, and always view k_F as a subfield of $\overline{\mathbb{F}}_p$ via this injection. For an element $x \in k_F$, we let $[x] \in \mathcal{O}_F$ denote its Teichmüller lift; conversely, for $y \in \mathcal{O}_F$, we let $\overline{y} \in k_F$ denote its reduction mod $\mathfrak{p}_F$. Finally, we let ε_F denote the composition

$$F^\times \xrightarrow{\mathrm{N}_{F/\mathbb{Q}_p}} \mathbb{Q}_p^\times \xrightarrow{x \mapsto x|x|_p} \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{F}_p^\times \hookrightarrow \overline{\mathbb{F}}_p^\times.$$

Let $G := \mathrm{SL}_2(F)$, and let I_1 denote the "upper-triangular mod p" pro-p-Iwahori subgroup. The assumption $p > 2e(F/\mathbb{Q}_p) + 1$ guarantees that I_1 is torsion-free (see [Laz65, §III.3.2.7]). Let T denote the diagonal maximal torus, with maximal compact subgroup T_0 and maximal pro-p subgroup T_1. We let B denote the upper triangular Borel subgroup; then the unique positive root of T with respect to B is given by the character

$$\alpha\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}\right) = x^2.$$

We let $u_\alpha : F \longrightarrow G$ denote the map

$$u_\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}$$

(and define $u_{-\alpha}(x)$ as the analogous lower triangular unipotent matrix).

Let α^* denote the simple affine root $(-\alpha, 1)$. We have the following elements of $N_G(T)$, whose images in the affine Weyl group give a set of Coxeter generators:

$$\widehat{s_\alpha} := \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad \text{and} \quad \widehat{s_{\alpha^*}} := \begin{pmatrix} 0 & -\varpi^{-1} \\ \varpi & 0\end{pmatrix}.$$

Recall that the pro-p-Iwahori–Hecke algebra $\mathcal{H}$ of G is generated by operators $T_{\widehat{s_\alpha}}, T_{\widehat{s_{\alpha^*}}}$ and T_t for $t \in T_0$ (or, equivalently, by $T_{\widehat{s_\alpha}}, T_{\widehat{s_{\alpha^*}}}$ and $T_{\alpha^\vee(x)}$ for $x \in \mathcal{O}_F^\times$, where $\alpha^\vee(x) = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)$), subject to quadratic relations and braid relations. The purpose of this note is to compute the action of this algebra on some of the cohomology spaces $\mathrm{H}^i(I_1, \overline{\mathbb{F}}_p)$.

0.2. **Simple $\mathcal{H}$-modules.** We recall the classification of simple right $\mathcal{H}$-modules. Any simple right $\mathcal{H}$-module is isomorphic to one of the modules below, and there are no isomorphisms between any modules with distinct parameters.

- **Trivial character:** let χ_{triv} denote the one-dimensional module defined by the character
$$T_{\widehat{s_\alpha}} \longmapsto 0, \quad T_{\widehat{s_{\alpha^*}}} \longmapsto 0, \quad T_{\alpha^\vee(x)} \longmapsto 1,$$
where $x \in \mathcal{O}_F^\times$.
- **Sign character:** let χ_{sign} denote the one-dimensional module defined by the character
$$T_{\widehat{s_\alpha}} \longmapsto -1, \quad T_{\widehat{s_{\alpha^*}}} \longmapsto -1, \quad T_{\alpha^\vee(x)} \longmapsto 1,$$
where $x \in \mathcal{O}_F^\times$.
- **Principal series:** let $\chi : T \longrightarrow \overline{\mathbb{F}}_p^\times$ denote a smooth character. We have a process of parabolic induction, denoted $\mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\chi)$, which gives a two-dimensional right $\mathcal{H}$-module. Explicitly, if we let $\{v_1, v_2\}$ denote a basis, then the actions of the generators are given by
$$\begin{array}{rclrcl} v_1 \cdot T_{\widehat{s_\alpha}} & = & v_2 & v_2 \cdot T_{\widehat{s_\alpha}} & = & -\delta_{\chi|_{T_0},1} v_2 \\ v_1 \cdot T_{\widehat{s_{\alpha^*}}} & = & -\delta_{\chi|_{T_0},1} v_1 & v_2 \cdot T_{\widehat{s_{\alpha^*}}} & = & \chi\left(\left(\begin{smallmatrix} \varpi^{-1} & 0 \\ 0 & \varpi\end{smallmatrix}\right)\right) v_1 \\ v_1 \cdot T_t & = & \chi(t)^{-1} v_1 & v_2 \cdot T_t & = & \chi(t) v_2 \end{array}$$
When $\chi \neq 1$, the module $\mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\chi)$ is simple.

- **Supersingular modules:** let $0 \leq i \leq q-1$, and let $\mathfrak{ss}_i$ denote the one-dimensional module defined by the character
$$T_{\widehat{s_\alpha}} \longmapsto -\delta_{i,q-1}, \quad T_{\widehat{s_{\alpha^*}}} \longmapsto -\delta_{i,0}, \quad T_{\alpha^\vee(x)} \longmapsto \overline{x}^{\,-i},$$
where $x \in \mathcal{O}_F^\times$.

For future reference, we also note that for any right $\mathcal{H}$-module $\mathfrak{m}$, we may form the dual space $\mathfrak{m}^\vee$, equipped with a right action given by
$$(f \cdot T_g)(m) = f(m \cdot T_{g^{-1}}),$$
where $m \in \mathfrak{m}, f \in \mathfrak{m}^\vee$. For the simple modules above, we have
$$\chi_{\mathrm{triv}}^\vee \cong \chi_{\mathrm{triv}}, \qquad \chi_{\mathrm{sign}}^\vee \cong \chi_{\mathrm{sign}}, \qquad \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\chi)^\vee \cong \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}((\chi^{-1})^{s_\alpha}) \cong \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\chi),$$
$$\mathfrak{ss}_i^\vee \cong \begin{cases} \mathfrak{ss}_i & \text{if } i = 0, q-1, \\ \mathfrak{ss}_{q-1-i} & \text{if } 0 < i < q-1. \end{cases}$$
(For the case of irreducible parabolic induction, see [Abe19, Thm. 4.9].)

0.3. **Cohomology – preliminary.** We begin to consider cohomology spaces. By unwinding definitions, we have

$$\mathrm{H}^0(I_1, \overline{\mathbb{F}}_p) \cong \chi_{\mathrm{triv}}. \tag{H^0}$$

Consequently, by [Koz18, Thm. 7.1], we get

$$\mathrm{H}^{3[F:\mathbb{Q}_p]}(I_1, \overline{\mathbb{F}}_p) \cong \chi_{\mathrm{triv}}. \tag{$\mathrm{H}^{\mathrm{top}}$}$$

Recall from [Koz18, Lem. 5.1] that
$$I_1^{\mathrm{ab}} = u_\alpha(\mathcal{O}_F/\mathfrak{p}_F) \oplus u_{-\alpha}(\mathfrak{p}_F/\mathfrak{p}_F^2),$$
so that

$$\mathrm{H}^1(I_1, \overline{\mathbb{F}}_p) = \mathrm{span}\{\eta_{\alpha,r},\ \eta_{\alpha^*,r}\}_{0 \leq r \leq f-1}, \tag{1}$$

where
$$\eta_{\alpha,r}\left(\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix}\right) = \overline{b}^{p^r} \quad \text{and} \quad \eta_{\alpha^*,r}\left(\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix}\right) = \overline{c}^{p^r}$$
$(a,b,c,d \in \mathcal{O}_F)$. By [Koz18, Thm. 6.4], as an $\mathcal{H}$-module we have

$$\mathrm{H}^1(I_1, \overline{\mathbb{F}}_p) \cong \begin{cases} \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\varepsilon_{\mathbb{Q}_p} \circ \alpha) & \text{if } F = \mathbb{Q}_p, \\ \bigoplus_{r=0}^{f-1} \mathfrak{ss}_{2p^r} \oplus \mathfrak{ss}_{q-1-2p^r} & \text{if } F \neq \mathbb{Q}_p. \end{cases} \tag{H^1}$$

Consequently, by [Koz18, Thm. 7.2], we have

$$\mathrm{H}^{3[F:\mathbb{Q}_p]-1}(I_1, \overline{\mathbb{F}}_p) \cong \begin{cases} \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\varepsilon_{\mathbb{Q}_p} \circ \alpha) & \text{if } F = \mathbb{Q}_p, \\ \bigoplus_{r=0}^{f-1} \mathfrak{ss}_{2p^r} \oplus \mathfrak{ss}_{q-1-2p^r} & \text{if } F \neq \mathbb{Q}_p. \end{cases} \tag{$\mathrm{H}^{\mathrm{top}-1}$}$$

To proceed further, we examine I_1 in relation to other subgroups.

0.4. **Cohomology – congruence subgroups.** We let K and K^* denote the maximal compact subgroups associated to the reflections s_α and s_{α^*}, respectively, so that
$$K = \mathrm{SL}_2(\mathcal{O}_F) \qquad \text{and} \qquad K^* = \begin{pmatrix} \mathcal{O}_F & \mathfrak{p}_F^{-1} \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix} \cap G.$$
We let K_1 and K_1^* denote their first congruence subgroups, so that
$$K_1 = \begin{pmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{pmatrix} \cap G \quad \text{and} \quad K_1^* = \begin{pmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{pmatrix} \cap G.$$
We have $K_1 = I_1 \cap \widehat{s_\alpha} I_1 \widehat{s_\alpha}^{-1}$ and $K_1^* = I_1 \cap \widehat{s_{\alpha^*}} I_1 \widehat{s_{\alpha^*}}^{-1}$.

We do the calculations for K_1; the calculations for K_1^* follow by conjugation. One can compute in a straightforward way that
$$K_1^{\mathrm{ab}} = u_{-\alpha}(\mathfrak{p}_F/\mathfrak{p}_F^2) \oplus T_1/T_2 \oplus u_\alpha(\mathfrak{p}_F/\mathfrak{p}_F^2).$$
Therefore,
$$\mathrm{H}^1(K_1, \overline{\mathbb{F}}_p) = \mathrm{span}\{\eta_{\mathrm{u},r},\ \eta_{\mathrm{d},r},\ \eta_{\mathrm{l},r}\}_{0 \leq r \leq f-1},$$

where

$$\eta_{\mathrm{u},r}\left(\begin{pmatrix}1+\varpi a & \varpi b\\ \varpi c & 1+\varpi d\end{pmatrix}\right)=\overline{b}^{p^r},\quad \eta_{\mathrm{d},r}\left(\begin{pmatrix}1+\varpi a & \varpi b\\ \varpi c & 1+\varpi d\end{pmatrix}\right)=\overline{a}^{p^r},\quad \eta_{\mathrm{l},r}\left(\begin{pmatrix}1+\varpi a & \varpi b\\ \varpi c & 1+\varpi d\end{pmatrix}\right)=\overline{c}^{p^r}$$

$(a,b,c,d\in\mathcal{O}_F)$. We also have

$$\mathrm{H}^1(K_1^*,\overline{\mathbb{F}}_p)=\mathrm{span}\{\eta^*_{\mathrm{u},r},\ \eta^*_{\mathrm{d},r},\ \eta^*_{\mathrm{l},r}\}_{0\le r\le f-1},$$

where the starred homomorphisms are defined similarly.

The group K acts by conjugation on $\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)$, and we have

$$\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)\cong\bigoplus_{r=0}^{f-1}\mathrm{Sym}^2(\overline{\mathbb{F}}_p^{\oplus 2})^{\mathrm{Fr}^r} \tag{2}$$

as K-representations (and similarly for K^*; see [BP12, Prop. 5.1]).

Finally, if F is unramified over $\mathbb{Q}_p$, then the dimension of $\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)$ is equal to the dimension of K_1 as a p-adic manifold, and therefore K_1 is uniform (likewise for K_1^*; see [KS14, Prop. 1.10, Rmk. 1.11]). We then obtain

$$\mathrm{H}^i(K_1,\overline{\mathbb{F}}_p)\cong\bigwedge^i\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)$$

([SW00, Thm. 5.1.5]).

0.5. **Cohomology – quotients.** The quotients I_1/K_1 and I_1/K_1^* are both isomorphic to $\mathcal{O}_F/\mathfrak{p}_F\cong\mathbb{F}_p^f$ as abelian groups. By the Künneth formula, we have

$$\mathrm{H}^i(I_1/K_1,\overline{\mathbb{F}}_p)\cong\bigoplus_{i_1+\ldots+i_f=i}\mathrm{H}^{i_1}(\mathbb{F}_p,\overline{\mathbb{F}}_p)\otimes\cdots\otimes\mathrm{H}^{i_f}(\mathbb{F}_p,\overline{\mathbb{F}}_p). \tag{3}$$

We can write some low-degree terms explicitly. Since $\mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p)\cong\mathrm{Hom}(I_1/K_1,\overline{\mathbb{F}}_p)$, we have

$$\mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p)=\mathrm{span}\{\overline{\eta}_r\}_{0\le r\le f-1}, \tag{4}$$

where

$$\overline{\eta}_r\left(\begin{pmatrix}1 & \overline{b}\\ 0 & 1\end{pmatrix}\right)=\overline{b}^{p^r},$$

$(b\in\mathcal{O}_F)$. We write

$$\mathrm{H}^1(I_1/K_1^*,\overline{\mathbb{F}}_p)=\mathrm{span}\{\overline{\eta}^*_r\}_{0\le r\le f-1},$$

where $\overline{\eta}^*_r$ are defined similarly (on lower-triangular matrices).

Given $0\le r<s\le f-1$, we can form the cup products $\overline{\eta}_r\smile\overline{\eta}_s\in\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)$. It is easy to check $\overline{\eta}_r\smile\overline{\eta}_s\neq 0$, and that the set

$$\{\overline{\eta}_r\smile\overline{\eta}_s\}_{0\le r<s\le f-1}$$

is linearly independent. The span of this set makes up the "$\mathrm{H}^1\otimes\mathrm{H}^1$ parts" of (3) above for $n=2$ (but the image of the element $\overline{\eta}_r\smile\overline{\eta}_s$ in the right-hand side of (3) is *not* a pure tensor).

To get the "H^2 parts" of (3) for $n=2$ above, we use the following construction. Consider the short exact sequence of trivial I_1/K_1-modules

$$0\longrightarrow\mathbb{Z}/p\mathbb{Z}\cong p\mathbb{Z}/p^2\mathbb{Z}\longrightarrow\mathbb{Z}/p^2\mathbb{Z}\longrightarrow\mathbb{Z}/p\mathbb{Z}\longrightarrow 0,$$

and the associated long exact sequence of cohomology, with connecting homomorphism β (the first row with H^0's is exact):

$$0\longrightarrow\mathrm{H}^1(I_1/K_1,\mathbb{F}_p)\longrightarrow\mathrm{H}^1(I_1/K_1,\mathbb{Z}/p^2\mathbb{Z})\longrightarrow\mathrm{H}^1(I_1/K_1,\mathbb{F}_p)\xrightarrow{\beta}\mathrm{H}^2(I_1/K_1,\mathbb{F}_p)$$

Since I_1/K_1 annihilated by p, any homomorphism $I_1/K_1\longrightarrow\mathbb{Z}/p^2\mathbb{Z}$ factors through $p\mathbb{Z}/p^2\mathbb{Z}$, and consequently the first non-zero map is an isomorphism. Therefore, β is an injection. We may extend it linearly to $\beta:\mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p)\hookrightarrow\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)$. By dimension-counting, we conclude

$$\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)=\mathrm{span}\{\overline{\eta}_r\smile\overline{\eta}_s,\quad\beta(\overline{\eta}_t)\}_{0\le r<s\le f-1,0\le t\le f-1}. \tag{5}$$

0.6. **Second cohomology of I_1 — lower bound.**

0.6.1. *Inflations.* Combining (4), (1), and (2), we get

$$\begin{aligned}\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p)\right) &= f,\\ \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)\right) &= 2f,\\ \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)^{I_1/K_1}\right) &= f.\end{aligned}$$

The Hochschild–Serre spectral sequence gives a five-term exact sequence

$$0\longrightarrow \mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p)\longrightarrow \mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)\longrightarrow \mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)^{I_1/K_1}\longrightarrow \mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)\longrightarrow \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p),$$

and the dimension calculations imply that the transgression map $\mathrm{H}^1(K_1,\overline{\mathbb{F}}_p)^{I_1/K_1}\longrightarrow \mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)$ is 0. Therefore, the inflation map

$$\mathrm{inf}^{I_1}_{I_1/K_1}:\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)\longrightarrow \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)$$

is injective (and likewise for the group K_1^*). Moreover, once can check (using, e.g., the eigenvalues of the conjugation action of the elements $\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathcal{O}_F)$, $a\in\mathcal{O}_F^\times$) that the images of $\mathrm{inf}^{I_1}_{I_1/K_1}$ and $\mathrm{inf}^{I_1}_{I_1/K_1^*}$ intersect trivially. Therefore, we get an inclusion

$$\mathrm{inf}^{I_1}_{I_1/K_1}\left(\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)\right)\ \oplus\ \mathrm{inf}^{I_1}_{I_1/K_1^*}\left(\mathrm{H}^2(I_1/K_1^*,\overline{\mathbb{F}}_p)\right)\subset \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p). \tag{6}$$

We simplify the expression (6). Let $\beta:\mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)\hookrightarrow \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)$ denote the Bockstein map of I_1 (since I_1^{ab} is annihilated by p, β is injective). Since β is defined as a differential, [NSW08, Prop. 1.5.2] implies we have a commutative diagram of $\overline{\mathbb{F}}_p$-vector spaces:

$$\begin{array}{ccc}\mathrm{H}^1(I_1/K_1,\overline{\mathbb{F}}_p) & \xhookrightarrow{\mathrm{inf}^{I_1}_{I_1/K_1}} & \mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)\\ \big\downarrow\beta & & \big\downarrow\beta\\ \mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p) & \xhookrightarrow{\mathrm{inf}^{I_1}_{I_1/K_1}} & \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)\end{array}$$

Thus, applying $\mathrm{inf}^{I_1}_{I_1/K_1}$ to (5) gives

$$\begin{aligned}\mathrm{inf}^{I_1}_{I_1/K_1}\left(\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)\right) &= \mathrm{span}\left\{\mathrm{inf}^{I_1}_{I_1/K_1}(\overline{\eta}_r\smile\overline{\eta}_s),\quad \mathrm{inf}^{I_1}_{I_1/K_1}(\beta(\overline{\eta}_t))\right\}_{0\le r<s\le f-1,0\le t\le f-1}\\ &= \mathrm{span}\left\{\mathrm{inf}^{I_1}_{I_1/K_1}(\overline{\eta}_r)\smile \mathrm{inf}^{I_1}_{I_1/K_1}(\overline{\eta}_s),\quad \beta(\mathrm{inf}^{I_1}_{I_1/K_1}(\overline{\eta}_t))\right\}_{0\le r<s\le f-1,0\le t\le f-1}\\ &= \mathrm{span}\left\{\eta_{\alpha,r}\smile\eta_{\alpha,s},\quad \beta(\eta_{\alpha,t})\right\}_{0\le r<s\le f-1,0\le t\le f-1}\end{aligned}$$

In particular, the injectivity of the inflation maps implies that the above spanning set is linearly independent. Proceeding likewise with K_1^*, we conclude that the following set is linearly independent:

$$\left\{\eta_{\alpha,r}\smile\eta_{\alpha,s},\quad \beta(\eta_{\alpha,t}),\quad \eta_{\alpha^*,r}\smile\eta_{\alpha^*,s},\quad \beta(\eta_{\alpha^*,t})\right\}_{0\le r<s\le f-1,0\le t\le f-1}$$

0.6.2. *More cup products.* We now consider cup products of the form $\eta_{\alpha,r}\smile\eta_{\alpha^*,s}$ for $0\le r,s\le f-1$.

Lemma 0.1. *We have $\eta_{\alpha,r}\smile\eta_{\alpha^*,s}\neq 0$ if and only if $r\neq s$.*

Proof. Suppose that there exists a 1-cochain $\psi: I_1\longrightarrow\overline{\mathbb{F}}_p$ such that $d\psi=\eta_{\alpha,r}\smile\eta_{\alpha^*,s}$; that is, suppose we have

$$\psi(h_1)+\psi(h_2)-\psi(h_1h_2)=\eta_{\alpha,r}(h_1)\eta_{\alpha^*,s}(h_2) \tag{7}$$

for $h_1,h_2\in I_1$. The right-hand side is 0 if $h_1\in B^-\cap I_1$ or $h_2\in B\cap I_1$. In particular, ψ is a homomorphism when restricted to $B\cap I_1$ or $B^-\cap I_1$. Thus, we have

$$\begin{aligned}\psi\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right) &= \nu\overline{b}^{p^m}\\ \psi\left(\begin{pmatrix}1 & 0\\ \varpi c & 1\end{pmatrix}\right) &= \lambda\overline{c}^{p^\ell},\end{aligned}$$

where $\nu,\lambda\in\overline{\mathbb{F}}_p$, $b,c\in\mathcal{O}_F$, and $0\le \ell,m\le f-1$. Therefore, by the Iwahori decomposition, we have

$$\psi\left(\begin{pmatrix}1+\varpi a & b\\ \varpi c & 1+\varpi d\end{pmatrix}\right) = \psi\left(\begin{pmatrix}1 & 0\\ \varpi c(1+\varpi a)^{-1} & 1\end{pmatrix}\begin{pmatrix}1+\varpi a & b\\ 0 & (1+\varpi a)^{-1}\end{pmatrix}\right)$$

$$\begin{aligned}
&\overset{(7)}{=} \psi\left(\begin{pmatrix}1 & 0\\ \varpi c(1+\varpi a)^{-1} & 1\end{pmatrix}\right) + \psi\left(\begin{pmatrix}1+\varpi a & b\\ 0 & (1+\varpi a)^{-1}\end{pmatrix}\right)\\
&\overset{(7)}{=} \psi\left(\begin{pmatrix}1 & 0\\ \varpi c(1+\varpi a)^{-1} & 1\end{pmatrix}\right) + \psi\left(\begin{pmatrix}1+\varpi a & 0\\ 0 & (1+\varpi a)^{-1}\end{pmatrix}\right) + \psi\left(\begin{pmatrix}1 & b(1+\varpi a)^{-1}\\ 0 & 1\end{pmatrix}\right)\\
&= \lambda\overline{c}^{p^\ell} + (\psi\circ\alpha^\vee)(1+\varpi a) + \nu\overline{b}^{p^m}. \qquad (8)
\end{aligned}$$

Next, suppose $h_1 = \left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix}1 & 0\\ \varpi & 1\end{smallmatrix}\right)$. Using (8), the left-hand-side of (7) becomes

$$\psi\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\right)+\psi\left(\begin{pmatrix}1 & 0\\ \varpi & 1\end{pmatrix}\right)-\psi\left(\begin{pmatrix}1+\varpi a & a\\ \varpi & 1\end{pmatrix}\right) = \nu\overline{a}^{p^m}+\lambda-\left(\lambda+(\psi\circ\alpha^\vee)(1+\varpi a)+\nu\overline{a}^{p^m}\right) = -(\psi\circ\alpha^\vee)(1+\varpi a),$$

while the right-hand side becomes

$$\eta_{\alpha,r}\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\right)\eta_{\alpha^*,s}\left(\begin{pmatrix}1 & 0\\ \varpi & 1\end{pmatrix}\right) = \overline{a}^{p^r}.$$

On the other hand, taking $h_1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix}1 & 0\\ \varpi a & 1\end{smallmatrix}\right)$, the left-hand side of (7) becomes

$$\psi\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)+\psi\left(\begin{pmatrix}1 & 0\\ \varpi a & 1\end{pmatrix}\right)-\psi\left(\begin{pmatrix}1+\varpi a & 1\\ \varpi a & 1\end{pmatrix}\right) = \nu+\lambda\overline{a}^{p^\ell}-\left(\lambda\overline{a}^{p^\ell}+(\psi\circ\alpha^\vee)(1+\varpi a)+\nu\right) = -(\psi\circ\alpha^\vee)(1+\varpi a),$$

while the right-hand side becomes

$$\eta_{\alpha,r}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\eta_{\alpha^*,s}\left(\begin{pmatrix}1 & 0\\ \varpi a & 1\end{pmatrix}\right) = \overline{a}^{p^s}.$$

Collecting terms, we arrive at

$$\overline{a}^{p^s} = -(\psi\circ\alpha^\vee)(1+\varpi a) = \overline{a}^{p^r},$$

which forces $r = s$.

Conversely, if $r = s$, then the function

$$\psi\left(\begin{pmatrix}1+\varpi a & b\\ \varpi c & 1+\varpi d\end{pmatrix}\right) = -\overline{a}^{p^r}$$

satisfies the equation (7) for all $h_1, h_2 \in I_1$, which implies $\eta_{\alpha,r} \smile \eta_{\alpha^*,r} = 0$. □

The action on $\eta_{\alpha,r} \smile \eta_{\alpha^*,s}$ of $T_{\alpha^\vee(x)} = \alpha^\vee(x)_*^{-1}$ for $x \in \mathcal{O}_F^\times$ is given by the scalar $\overline{x}^{2p^r-2p^s}$. We therefore see that the set $\{\eta_{\alpha,r} \smile \eta_{\alpha^*,s}\}_{0\le r\ne s\le f-1}$ is linearly independent, and its span intersects

$$\mathrm{inf}^{I_1}_{I_1/K_1}\left(\mathrm{H}^2(I_1/K_1,\overline{\mathbb{F}}_p)\right) \oplus \mathrm{inf}^{I_1}_{I_1/K_1^*}\left(\mathrm{H}^2(I_1/K_1^*,\overline{\mathbb{F}}_p)\right)$$

trivially. Thus, the following set of vectors is linearly independent:

$$\{\eta_{\alpha,r}\smile\eta_{\alpha,s},\quad \beta(\eta_{\alpha,t}),\quad \eta_{\alpha^*,r}\smile\eta_{\alpha^*,s},\quad \beta(\eta_{\alpha^*,t})\}_{0\le r<s\le f-1, 0\le t\le f-1} \cup \quad \{\eta_{\alpha,r}\smile\eta_{\alpha^*,s}\}_{0\le r\ne s\le f-1} \qquad (9)$$

In particular, we obtain the bound

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)\right) \ge 2f^2. \qquad (10)$$

0.7. **Hecke action.** Finally, we calculate the action of $\mathcal{H}$ on $\mathrm{span}_{\overline{\mathbb{F}}_p}\{(9)\}$.

Note first that the span of the elements $\beta(\eta_{\alpha,t})$ and $\beta(\eta_{\alpha^*,t})$ is simply the image of $\beta : \mathrm{H}^1(I_1,\overline{\mathbb{F}}_p) \hookrightarrow \mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)$. Since β is defined as a differential corresponding to a short exact sequence of I_1-modules, it commutes with restriction, corestriction, conjugation, and inflation ([NSW08, Prop. 1.5.2]). Recall that if $\varphi \in \mathrm{H}^i(I_1,\overline{\mathbb{F}}_p)$, then

$$\varphi\cdot T_g = \mathrm{cor}^{I_1}_{I_1\cap g^{-1}I_1g}\circ g_*^{-1}\circ \mathrm{res}^{I_1}_{I_1\cap gI_1g^{-1}}(\varphi).$$

Thus, we see that β is in fact $\mathcal{H}$-equivariant. In particular, we have $\beta(\mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)) \cong \mathrm{H}^1(I_1,\overline{\mathbb{F}}_p)$ as right $\mathcal{H}$-modules, and we know the structure of the latter space (it is entirely supersingular as soon as $F \ne \mathbb{Q}_p$). Hence,

$$\mathrm{span}_{\overline{\mathbb{F}}_p}\{\beta(\eta_{\alpha,t}),\beta(\eta_{\alpha^*,t})\}_{0\le t\le f-1} \cong \begin{cases}\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_T}(\varepsilon_{\mathbb{Q}_p}\circ\alpha) & \text{if } F = \mathbb{Q}_p,\\ \bigoplus_{r=0}^{f-1}\mathfrak{ss}_{2p^r}\oplus\mathfrak{ss}_{q-1-2p^r} & \text{if } F \ne \mathbb{Q}_p.\end{cases} \qquad (11)$$

Next, we assume $f \geq 2$. By [NSW08, Prop. 1.5.3], the cup product commutes with restriction, conjugation, and inflation (but *not* corestriction). Consequently, if $\varphi \in \mathrm{H}^i(I_1, \overline{\mathbb{F}}_p)$ and $\psi \in \mathrm{H}^j(I_1, \overline{\mathbb{F}}_p)$, we have

$$(\varphi \smile \psi) \cdot T_g = \mathrm{cor}^{I_1}_{I_1 \cap g^{-1} I_1 g}\left(g_*^{-1} \circ \mathrm{res}^{I_1}_{I_1 \cap g I_1 g^{-1}}(\varphi) \smile g_*^{-1} \circ \mathrm{res}^{I_1}_{I_1 \cap g I_1 g^{-1}}(\psi)\right).$$

We note that

$$\mathrm{res}^{I_1}_{I_1 \cap \widehat{s_\alpha} I_1 \widehat{s_\alpha}^{-1}}(\eta_{\alpha,r}) = \mathrm{res}^{I_1}_{K_1}(\eta_{\alpha,r}) = 0$$

and

$$\mathrm{res}^{I_1}_{I_1 \cap \widehat{s_{\alpha^*}} I_1 \widehat{s_{\alpha^*}}^{-1}}(\eta_{\alpha^*,r}) = \mathrm{res}^{I_1}_{K_1^*}(\eta_{\alpha^*,r}) = 0,$$

which gives

$$(\eta_{\alpha,r} \smile \psi) \cdot T_{\widehat{s_\alpha}} = 0, \qquad (\varphi \smile \eta_{\alpha^*,s}) \cdot T_{\widehat{s_{\alpha^*}}} = 0. \tag{12}$$

The equation (12) implies that each $\eta_{\alpha,r} \smile \eta_{\alpha^*,s}$ gives a one-dimensional supersingular $\mathcal{H}$-module: the operators $T_{\widehat{s_\alpha}}$ and $T_{\widehat{s_{\alpha^*}}}$ act by 0, while $T_{\alpha^\vee(x)}$ acts by $\overline{x}^{2p^r - 2p^s}$. Thus,

$$\mathrm{span}_{\overline{\mathbb{F}}_p}\{\eta_{\alpha,r} \smile \eta_{\alpha^*,s}\}_{0 \leq r \neq s \leq f-1} \cong \bigoplus_{0 \leq r \neq s \leq f-1} \mathfrak{ss}_{[-2p^r + 2p^s]}, \tag{13}$$

where $[i]$ denotes the unique element of $\{0, \ldots, q-2\}$ congruent to i modulo $q-1$.

Finally, we consider the $\mathcal{H}$-module generated by $\eta_{\alpha,r} \smile \eta_{\alpha,s}$.

Lemma 0.2. *If $f \geq 2$, we have*

$$\mathrm{span}_{\overline{\mathbb{F}}_p}\{\eta_{\alpha,r} \smile \eta_{\alpha,s},\ \eta_{\alpha^*,r} \smile \eta_{\alpha^*,s}\}_{0 \leq r < s \leq f-1} = \begin{cases} \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_T}(\varepsilon_{\mathbb{Q}_{p^2}} \circ \alpha) & \text{if } F = \mathbb{Q}_{p^2}, \\ \bigoplus_{0 \leq r < s \leq f-1} \mathfrak{ss}_{q-1-2p^r-2p^r} \oplus \mathfrak{ss}_{2p^r+2p^s} & \text{if } F \neq \mathbb{Q}_{p^2}. \end{cases} \tag{14}$$

Proof. We have $(\eta_{\alpha,r} \smile \eta_{\alpha,s}) \cdot T_{\widehat{s_\alpha}} = 0$, and the action of $T_{\alpha^\vee(x)}$ is given by the scalar $\overline{x}^{2p^r + 2p^s}$. Therefore it suffices to calculate the action of $T_{\widehat{s_{\alpha^*}}}$. We have

$$\begin{aligned}
(\eta_{\alpha,r} \smile \eta_{\alpha,s}) \cdot T_{\widehat{s_{\alpha^*}}} &= \mathrm{cor}^{I_1}_{K_1^*}\left((\widehat{s_{\alpha^*}}^{-1})_* \circ \mathrm{res}^{I_1}_{K_1^*}(\eta_{\alpha,r}) \smile (\widehat{s_{\alpha^*}}^{-1})_* \circ \mathrm{res}^{I_1}_{K_1^*}(\eta_{\alpha,s})\right) \\
&= \mathrm{cor}^{I_1}_{K_1^*}\left((\widehat{s_{\alpha^*}}^{-1})_* \eta^*_{\mathrm{u},r} \smile (\widehat{s_{\alpha^*}}^{-1})_* \eta^*_{\mathrm{u},s}\right) \\
&= \mathrm{cor}^{I_1}_{K_1^*}\left((-\eta^*_{\mathrm{l},r}) \smile (-\eta^*_{\mathrm{l},s})\right) \\
&= \mathrm{cor}^{I_1}_{K_1^*}\left(\eta^*_{\mathrm{l},r} \smile \eta^*_{\mathrm{l},s}\right).
\end{aligned}$$

Given $h = \left(\begin{smallmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{smallmatrix}\right) \in I_1$, we define $r(h) := u_{-\alpha}(\varpi[\overline{c}])$, so that $hr(h)^{-1} \in K_1^*$. Unwinding definitions in [NSW08], we see that an inhomogenous 2-cocycle representing $\mathrm{cor}^{I_1}_{K_1^*}(\eta^*_{\mathrm{l},r} \smile \eta^*_{\mathrm{l},s})$ is given by

$$\begin{aligned}
(h_1, h_2) \longmapsto \sum_{x \in k_F} &\eta^*_{\mathrm{l},r}\left(u_{-\alpha}(\varpi[x]) h_1 r(u_{-\alpha}(\varpi[x]) h_1)^{-1}\right) \\
&\cdot \eta^*_{\mathrm{l},s}\left(r(u_{-\alpha}(\varpi[x]) h_1) h_1^{-1} u_{-\alpha}(\varpi[x])^{-1} \cdot u_{-\alpha}(\varpi[x]) h_1 h_2 r(u_{-\alpha}(\varpi[x]) h_1 h_2)^{-1}\right) \\
= \sum_{x \in k_F} &\eta^*_{\mathrm{l},r}\left(u_{-\alpha}(\varpi[x]) h_1 r(u_{-\alpha}(\varpi[x]) h_1)^{-1}\right) \cdot \eta^*_{\mathrm{l},s}\left(r(u_{-\alpha}(\varpi[x]) h_1) h_2 r(u_{-\alpha}(\varpi[x]) h_1 h_2)^{-1}\right)
\end{aligned}$$

We evaluate some terms in this sum. Note first that

$$u_{-\alpha}(\varpi[x]) h_1 r(u_{-\alpha}(\varpi[x]) h_1)^{-1} =$$

$$\begin{pmatrix} 1 + \varpi(a_1 - b_1[x + \overline{c_1}]) & b_1 \\ \varpi([x] + c_1 - [x + \overline{c_1}]) + \varpi^2(a_1[x] - d_1[x + \overline{c_1}] - b_1[x^2 + \overline{c_1}x]) & 1 + \varpi(d_1 + b_1[x]) \end{pmatrix}$$

$$r(u_{-\alpha}(\varpi[x]) h_1) h_2 r(u_{-\alpha}(\varpi[x]) h_1 h_2)^{-1} =$$

$$\begin{pmatrix} 1 + \varpi(a_2 - b_2[x + \overline{c_1} + \overline{c_2}]) & b_2 \\ \varpi([x + \overline{c_1}] + c_2 - [x + \overline{c_1} + \overline{c_2}]) + \varpi^2(a_2[x + \overline{c_1}] - d_2[x + \overline{c_1} + \overline{c_2}] - b_2[x + \overline{c_1}][x + \overline{c_1} + \overline{c_2}]) & 1 + \varpi(d_2 + b_2[x + \overline{c_1}]) \end{pmatrix}$$

Thus, the sum above becomes

$$\sum_{x\in k_F}\left(\varpi^{-1}\Big([x]+c_1-[x+\overline{c_1}]\Big)+\overline{a_1}x-\overline{d_1}(x+\overline{c_1})-\overline{b_1}(x^2+\overline{c_1}x)\right)^{p^r}$$
$$\cdot\left(\varpi^{-1}\Big([x+\overline{c_1}]+c_2-[x+\overline{c_1}+\overline{c_2}]\Big)+\overline{a_2}(x+\overline{c_1})-\overline{d_2}(x+\overline{c_1}+\overline{c_2})-\overline{b_2}(x+\overline{c_1})(x+\overline{c_1}+\overline{c_2})\right)^{p^s}.$$

We now analyze some Witt vector calculations in greater depth. Suppose $z\in k_F$ and $c\in\mathcal{O}_F$. We have

$$[z]+c-[z+\overline{c}]=(c-[\overline{c}])+p\left[\sum_{k=1}^{p-1}\overline{-\binom{p}{k}p^{-1}}z^{k/p}\overline{c}^{(p-k)/p}\right]+\ldots,$$

where the ellipsis denote higher order terms in the Witt vector expansion. In particular, we see that if $F/\mathbb{Q}_p$ is ramified, then $p\varpi^{-1}\in\mathfrak{p}_F$, and the sum above reduces to

$$\sum_{x\in k_F}\left(\varpi^{-1}\Big(c_1-[\overline{c_1}]\Big)+\overline{a_1}x-\overline{d_1}(x+\overline{c_1})-\overline{b_1}(x^2+\overline{c_1}x)\right)^{p^r}$$
$$\cdot\left(\varpi^{-1}\Big(c_2-[\overline{c_2}]\Big)+\overline{a_2}(x+\overline{c_1})-\overline{d_2}(x+\overline{c_1}+\overline{c_2})-\overline{b_2}(x+\overline{c_1})(x+\overline{c_1}+\overline{c_2})\right)^{p^s}$$

Be expanding, we are left with a sum of terms of the form $\sum_{x\in k_F}\overline{a}x^{\delta_rp^r+\delta_sp^s}$, where $\delta_r,\delta_s\in\{0,1,2\}$. Since $p\geq 5$, we have $\delta_rp^r+\delta_sp^s<p^f-1$, and therefore all such terms must vanish.

We may therefore assume that $F/\mathbb{Q}_p$ is unramified (and take $\varpi=p$ to be our uniformizer). The above sum now becomes

$$\sum_{x\in k_F}\left(\varpi^{-1}\Big(c_1-[\overline{c_1}]\Big)+\sum_{k=1}^{p-1}\overline{-\binom{p}{k}p^{-1}}x^{k/p}\overline{c_1}^{(p-k)/p}+\overline{a_1}x-\overline{d_1}(x+\overline{c_1})-\overline{b_1}(x^2+\overline{c_1}x)\right)^{p^r}$$
$$\cdot\left(\varpi^{-1}\Big(c_2-[\overline{c_2}]\Big)+\sum_{k=1}^{p-1}\overline{-\binom{p}{k}p^{-1}}(x+\overline{c_1})^{k/p}\overline{c_2}^{(p-k)/p}+\overline{a_2}(x+\overline{c_1})-\overline{d_2}(x+\overline{c_1}+\overline{c_2})-\overline{b_2}(x+\overline{c_1})(x+\overline{c_1}+\overline{c_2})\right)^{p^s}$$

Expanding once again, we find a sum of terms of the form $\sum_{x\in k_F}\overline{a}x^{\delta_rp^r+\delta_sp^s}$, where now $\delta_r,\delta_s\in\{0,1,2,1/p,2/p,\ldots,(p-1)/p\}$. In order for such a sum to be nonzero, we must have $\delta_rp^r+\delta_sp^s\equiv 0 \pmod{p^f-1}$ and $(\delta_r,\delta_s)\neq(0,0)$. Examining possibilities, we see that if the sum is nonzero, then we must have $f=2,\delta_r=\delta_s=(p-1)/p$. (This also forces $r=0,s=1$.) In this case, the sum above becomes

$$\begin{aligned}\sum_{x\in k_F}\overline{-\binom{p}{p-1}p^{-1}}x^{(p-1)/p}\overline{c_1}^{1/p}\left(\overline{-\binom{p}{p-1}p^{-1}}x^{(p-1)/p}\overline{c_2}^{1/p}\right)^{p} &= \sum_{x\in k_F}\overline{c_1}^p\overline{c_2}x^{p^2-1}\\ &= -\overline{c_1}^p\overline{c_2}\\ &= -(\eta_{\alpha^*,1}\smile\eta_{\alpha^*,0})(h_1,h_2).\end{aligned}$$

Combining these calculations, we conclude

$$(\eta_{\alpha,r}\smile\eta_{\alpha,s})\cdot T_{\widehat{s_{\alpha^*}}}=\begin{cases}-(\eta_{\alpha^*,1}\smile\eta_{\alpha^*,0})=\eta_{\alpha^*,0}\smile\eta_{\alpha^*,1} & \text{if } F=\mathbb{Q}_{p^2},\\ 0 & \text{if } F\neq\mathbb{Q}_{p^2}.\end{cases}$$

Further, conjugating by $\left(\begin{smallmatrix}0&1\\ \varpi&0\end{smallmatrix}\right)$ shows that we have

$$(\eta_{\alpha^*,r}\smile\eta_{\alpha^*,s})\cdot T_{\widehat{s_\alpha}}=\begin{cases}\eta_{\alpha,0}\smile\eta_{\alpha,1} & \text{if } F=\mathbb{Q}_{p^2},\\ 0 & \text{if } F\neq\mathbb{Q}_{p^2}.\end{cases}$$

□

Remark 0.3. A similar calculation with cup products shows that when $F=\mathbb{Q}_{p^f}$, the element $\eta_{\alpha,0}\smile\eta_{\alpha,1}\smile\ldots\smile\eta_{\alpha,f-1}$ generates an $\mathcal{H}$-submodule of $\mathrm{H}^f(I_1,\overline{\mathbb{F}}_p)$ isomorphic to $\mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\varepsilon_{\mathbb{Q}_{p^f}}\circ\alpha)$.

Combining equations (11), (13), and (14), we arrive at

$$\text{(15)}\qquad \mathrm{span}_{\overline{\mathbb{F}}_p}\{(9)\} \cong \begin{cases} \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\varepsilon_{\mathbb{Q}_p}\circ\alpha) & \text{if } F = \mathbb{Q}_p,\\ \mathfrak{ss}_2 \oplus \mathfrak{ss}_{p-3} & \text{if } f = 1, F \neq \mathbb{Q}_p,\\ \big(\mathfrak{ss}_2 \oplus \mathfrak{ss}_{p^2-3} \oplus \mathfrak{ss}_{2p} \oplus \mathfrak{ss}_{p^2-1-2p}\big) \oplus \big(\mathfrak{ss}_{2p-2} \oplus \mathfrak{ss}_{p^2-1-2p+2}\big) & \\ \qquad\qquad \oplus \mathrm{Ind}_{\mathcal{H}_T}^{\mathcal{H}}(\varepsilon_{\mathbb{Q}_{p^2}}\circ\alpha) & \text{if } F = \mathbb{Q}_{p^2},\\ \Big(\bigoplus_{r=0}^{f-1} \mathfrak{ss}_{2p^r} \oplus \mathfrak{ss}_{q-1-2p^r}\Big) \oplus \Big(\bigoplus_{0\le r\neq s\le f-1} \mathfrak{ss}_{[-2p^r+2p^s]}\Big) & \\ \qquad \oplus \Big(\bigoplus_{0\le r<s\le f-1} \mathfrak{ss}_{q-1-2p^r-2p^r} \oplus \mathfrak{ss}_{2p^r+2p^s}\Big) & \text{if } f\ge 2, F\neq \mathbb{Q}_{p^2}.\end{cases}$$

0.8. **Second cohomology of I_1 — upper bound.** Our next task will be to use a spectral sequence to try to get an upper bound on the dimension of $\mathrm{H}^2(I_1, \overline{\mathbb{F}}_p)$. For simplicity, we assume that F is unramified over $\mathbb{Q}_p$ and take $\varpi = p$ to be our uniformizer.

We define a function $\omega : I_1 \longrightarrow \mathbb{R}_{>0}\cup\{\infty\}$ as follows:

$$\omega\left(\begin{pmatrix} 1+pa & b \\ pc & 1+pd\end{pmatrix}\right) := \min\left\{\mathrm{val}_p(a)+1, \quad \mathrm{val}_p(b)+\frac{1}{2}, \quad \mathrm{val}_p(c)+\frac{1}{2}, \quad \mathrm{val}_p(d)+1\right\}$$

By [LS24, Prop. 3.5], the function ω defines a p-valuation on I_1. Furthermore, by choosing a basis $\{x_r\}_{0\le r\le f-1}$ of $\mathcal{O}_F$ over $\mathbb{Z}_p$, we see that an ordered basis of I_1 is given by the elements

$$\{u_\alpha(x_r), \quad u_{-\alpha}(px_r), \quad \alpha^\vee(\exp(px_r))\}_{0\le r\le f-1}.$$

Let $\mathrm{gr}_\omega(I_1)$ denote the graded group associated to I_1 (and ω), and let $\mathfrak{I} := \mathrm{Lie}_\omega(I_1) := \mathrm{gr}_\omega(I_1)\otimes_{\mathbb{F}_p[P]}\overline{\mathbb{F}}_p$ denote the Lie algebra of I_1 associated to ω. Here, P denotes the operator which sends $hI_{1,\nu+}$ to $h^pI_{1,(\nu+1)+}$. The Lie algebra $\mathfrak{I}$ has a Lie bracket induced by the commutator in I_1. By decomposing with respect to field embeddings, we have an isomorphism of Lie algebras

$$\mathfrak{I} = \bigoplus_{r=0}^{f-1}\mathfrak{g}_r,$$

where $\mathfrak{g}_r$ is a 3-dimensional $\overline{\mathbb{F}}_p$-Lie algebra with basis e_r, f_r, h_r and bracket relations

$$[e_r, f_r] = h_r, \qquad [h_r, e_r] = 0, \qquad [h_r, f_r] = 0.$$

(The elements e_r (resp., f_r, resp., h_r) are linear combinations of the elements $\overline{u_\alpha(x_{r'})}\otimes 1$ (resp., $\overline{u_{-\alpha}(px_{r'})}\otimes 1$, resp., $\overline{\alpha^\vee(\exp(px_{r'}))}\otimes 1$).)

By [Sor21, Thm. 5.5], we have a convergent spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i,j}(\mathfrak{I}, \overline{\mathbb{F}}_p) \Longrightarrow \mathrm{H}^{i+j}(I_1, \overline{\mathbb{F}}_p)$$

Specializing to $\mathrm{H}^2(I_1, \overline{\mathbb{F}}_p)$, we obtain

$$\begin{aligned}\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^2(I_1,\overline{\mathbb{F}}_p)\right) &= \sum_{i+j=2}\dim_{\overline{\mathbb{F}}_p}(E_\infty^{i,j})\\ &\le \sum_{i+j=2}\dim_{\overline{\mathbb{F}}_p}(E_1^{i,j})\\ &= \sum_{i\in\mathbb{Z}}\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{i,2-i}(\mathfrak{I},\overline{\mathbb{F}}_p)\right).\end{aligned}\tag{16}$$

It therefore suffices to understand

$$\mathrm{H}^{i,2-i}(\mathfrak{I},\overline{\mathbb{F}}_p) = h^{i+(2-i)}\left(\mathrm{gr}^i(C^\bullet(\mathfrak{I},\overline{\mathbb{F}}_p))\right) = h^2\left(\mathrm{gr}^i(C^\bullet(\mathfrak{I},\overline{\mathbb{F}}_p))\right).$$

Here, $C^\bullet(\mathfrak{I},\overline{\mathbb{F}}_p)$ denotes the Chevalley–Eilenberg complex. The grading on this complex is defined as follows.

- We endow $\overline{\mathbb{F}}_p$ with the grading which puts $\overline{\mathbb{F}}_p$ in degree 0.
- The Lie algebra $\mathfrak{I}$ has a grading induced from the grading on $\mathrm{gr}_\omega(I_1)$. We have

$$\begin{aligned}\mathfrak{I} &= \mathfrak{I}^1\oplus\mathfrak{I}^2\\ &:= \mathrm{gr}_\omega^{1/2+\mathbb{Z}_{\ge 0}}(I_1)\otimes_{\mathbb{F}_p[P]}\overline{\mathbb{F}}_p \;\oplus\; \mathrm{gr}_\omega^{1+\mathbb{Z}_{\ge 0}}(I_1)\otimes_{\mathbb{F}_p[P]}\overline{\mathbb{F}}_p\\ &= \mathrm{span}_{\overline{\mathbb{F}}_p}\{e_r,\ f_r\}_{0\le r\le f-1} \;\oplus\; \mathrm{span}_{\overline{\mathbb{F}}_p}\{h_r\}_{0\le r\le f-1}.\end{aligned}$$

- For $j \geq 0$, the space $\bigwedge^j_{\overline{\mathbb{F}}_p} \mathfrak{J}$ is endowed with a grading as follows. Given homogeneous elements $v_1, \ldots, v_k \in \mathfrak{J}$, we let $\deg(v_1 \wedge \ldots \wedge v_k) = \sum_{\ell=1}^k \deg(v_\ell)$. We then set

$$\begin{aligned}\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p} \mathfrak{J} &= \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i\left(\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J}\right)\\ &:= \bigoplus_{i\in\mathbb{Z}} \mathrm{span}_{\overline{\mathbb{F}}_p}\left\{v \in \bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J} : \deg(v) = i\right\}.\end{aligned}$$

- We endow $\mathrm{Hom}_{\overline{\mathbb{F}}_p}(\bigwedge^j_{\overline{\mathbb{F}}_p} \mathfrak{J}, \overline{\mathbb{F}}_p)$ with a grading as follows:

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p\right) &= \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_{\overline{\mathbb{F}}_p}\left(\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p\right)\\ &:= \bigoplus_{i\in\mathbb{Z}}\left\{\mathtt{f} \in \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p\right) : \mathtt{f} \text{ is homogeneous of degree } i\right\}\\ &= \bigoplus_{i\in\mathbb{Z}_{\leq 0}} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\mathrm{gr}^{-i}\left(\bigwedge\nolimits^j_{\overline{\mathbb{F}}_p}\mathfrak{J}\right), \overline{\mathbb{F}}_p\right)\end{aligned}$$

- The Chevalley–Eilenberg complex $C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p)$ is defined by

$$0 \longrightarrow \overline{\mathbb{F}}_p \xrightarrow{\partial_1} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\mathfrak{J}, \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_2} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\bigwedge\nolimits^2_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_3} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\bigwedge\nolimits^3_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_4} \ldots$$

  The differentials are defined as follows: we have $\partial_1 = 0$ and given $j \geq 1$ and $\mathtt{f} \in \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\bigwedge^j_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p)$, we have

$$(\partial_{j+1}\mathtt{f})(X_1 \wedge X_2 \wedge \ldots \wedge X_{j+1}) = \sum_{1\leq a<b\leq j+1} (-1)^{a+b}\mathtt{f}([X_a, X_b] \wedge X_1 \wedge \ldots \wedge \widehat{X_a} \wedge \ldots \wedge \widehat{X_b} \wedge \ldots \wedge X_{j+1}).$$

- The differentials ∂_j respect the grading on $\mathrm{Hom}_{\overline{\mathbb{F}}_p}(\bigwedge^\bullet_{\overline{\mathbb{F}}_p}\mathfrak{J}, \overline{\mathbb{F}}_p)$, and therefore induce a complex $\mathrm{gr}^i(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p))$ given by

$$0 \longrightarrow \mathrm{gr}^i\left(\overline{\mathbb{F}}_p\right) \xrightarrow{\partial_1} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\mathrm{gr}^{-i}\left(\mathfrak{J}\right), \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_2} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\mathrm{gr}^{-i}\left(\bigwedge\nolimits^2_{\overline{\mathbb{F}}_p}\mathfrak{J}\right), \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_3} \mathrm{Hom}_{\overline{\mathbb{F}}_p}\left(\mathrm{gr}^{-i}\left(\bigwedge\nolimits^3_{\overline{\mathbb{F}}_p}\mathfrak{J}\right), \overline{\mathbb{F}}_p\right) \xrightarrow{\partial_4} \ldots$$

Recall that we are interested in calculating $h^2(\mathrm{gr}^i(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p)))$. We first note that $\mathrm{gr}^{-i}(\bigwedge^2_{\overline{\mathbb{F}}_p}\mathfrak{J}) \neq 0$ implies $i = -2, -3$, or -4. This gives

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{i,2-i}(\mathfrak{J}, \overline{\mathbb{F}}_p)\right) = \dim_{\overline{\mathbb{F}}_p}\left(h^2(\mathrm{gr}^i(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p)))\right) = 0 \qquad \text{if } i \notin \{-2, -3, -4\}. \tag{17}$$

We examine the remaining cases in turn.

0.8.1. $i = -2$. In this case, we note that $\mathrm{gr}^2(\bigwedge^3_{\overline{\mathbb{F}}_p}\mathfrak{J}) = 0$, so that $\partial_3 = 0$ in the complex $\mathrm{gr}^{-2}(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p))$. It suffices to understand the image of ∂_2. Suppose $\mathtt{f} \in \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\mathrm{gr}^2(\bigwedge^2_{\overline{\mathbb{F}}_p}\mathfrak{J}), \overline{\mathbb{F}}_p)$ is in the image of ∂_2. Then there exists $\mathtt{g} \in \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\mathrm{gr}^2(\mathfrak{J}), \overline{\mathbb{F}}_p)$ satisfying $\partial_2\mathtt{g} = \mathtt{f}$. In particular, if $X \wedge Y \in \mathfrak{J}^1 \wedge \mathfrak{J}^1 = \mathrm{gr}^2(\bigwedge^2_{\overline{\mathbb{F}}_p}\mathfrak{J})$, then

$$\mathtt{f}(X \wedge Y) = (\partial_2\mathtt{g})(X \wedge Y) = -\mathtt{g}([X, Y]).$$

Thus, $\mathtt{f}$ vanishes on $e_r \wedge e_s$ $(r < s)$, $f_r \wedge f_s$ $(r < s)$, and $e_r \wedge f_s$ $(r \neq s)$. The space of such homomorphisms is f-dimensional (being dual to the space spanned by $e_r \wedge f_r$ $(0 \leq r \leq f - 1)$), and therefore, we obtain

$$\begin{aligned}\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{-2,4}(\mathfrak{J}, \overline{\mathbb{F}}_p)\right) &= \dim_{\overline{\mathbb{F}}_p}\left(h^2(\mathrm{gr}^{-2}(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p)))\right)\\ &= \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{gr}^2\left(\bigwedge\nolimits^2_{\overline{\mathbb{F}}_p}\mathfrak{J}\right)\right) - \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{im}\left(\partial_2|_{\mathrm{gr}^{-2}(C^\bullet(\mathfrak{J}, \overline{\mathbb{F}}_p))}\right)\right)\\ &= \binom{2f}{2} - f\\ &= 2f^2 - 2f. \end{aligned} \tag{18}$$

(Note that this quantity is 0 if $f = 1$.)

0.8.2. $i = -3$. In this case, we note that $\mathrm{gr}^3(\mathfrak{I}) = 0$, so that $\partial_2 = 0$ in the complex $\mathrm{gr}^{-3}(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p))$. It therefore suffices to compute the kernel of ∂_3. If $\mathtt{f}$ lies in this kernel, and if $X, Y, Z \in \mathfrak{I}^1$, then we have $X \wedge Y \wedge Z \in \mathfrak{I}^1 \wedge \mathfrak{I}^1 \wedge \mathfrak{I}^1 = \mathrm{gr}^3(\bigwedge^3_{\overline{\mathbb{F}}_p} \mathfrak{I})$ and

$$0 = (\partial_3\mathtt{f})(X \wedge Y \wedge Z) = -\mathtt{f}([X,Y] \wedge Z) + \mathtt{f}([X,Z] \wedge Y) - \mathtt{f}([Y,Z] \wedge X).$$

In particular, we obtain the following relation:

$$\begin{aligned} \mathtt{f}(e_r \wedge h_s) &= -\mathtt{f}([e_s, f_s] \wedge e_r) \\ &= -\mathtt{f}([e_s, e_r] \wedge f_s) + \mathtt{f}([f_s, e_r] \wedge e_s) \\ &= \begin{cases} 0 & \text{if } r \neq s, \\ \mathtt{f}(-h_r \wedge e_r) & \text{if } r = s. \end{cases} \end{aligned}$$

Similarly, we have

$$\mathtt{f}(f_r \wedge h_s) = \begin{cases} 0 & \text{if } r \neq s, \\ -\mathtt{f}(h_r \wedge f_r) & \text{if } r = s. \end{cases}$$

Thus, $\mathtt{f}$ is determined by its values on the elements $e_r \wedge h_r$ and $f_r \wedge h_r$ $(0 \leq r \leq f-1)$. We therefore obtain

$$\begin{aligned} \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{-3,5}(\mathfrak{I}, \overline{\mathbb{F}}_p)\right) &= \dim_{\overline{\mathbb{F}}_p}\left(h^2(\mathrm{gr}^{-3}(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p)))\right) \\ &= \dim_{\overline{\mathbb{F}}_p}\left(\ker\left(\partial_3|_{\mathrm{gr}^{-2}(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p))}\right)\right) \\ &= 2f. \end{aligned} \tag{19}$$

0.8.3. $i = -4$. As with the previous case, we have $\mathrm{gr}^4(\mathfrak{I}) = 0$, so that $\partial_2 = 0$ in the complex $\mathrm{gr}^{-4}(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p))$, and it suffices to compute the kernel of ∂_3. If $\mathtt{f}$ lies in the kernel, and if $X \in \mathfrak{I}^1, Y \in \mathfrak{I}^1$ and $Z \in \mathfrak{I}^2$, then we have $X \wedge Y \wedge Z \in \mathrm{gr}^4(\bigwedge^3_{\overline{\mathbb{F}}_p} \mathfrak{I})$ and

$$\begin{aligned} 0 &= (\partial_3\mathtt{f})(X \wedge Y \wedge Z) \\ &= -\mathtt{f}([X,Y] \wedge Z) + \mathtt{f}([X,Z] \wedge Y) - \mathtt{f}([Y,Z] \wedge X) \\ &= -\mathtt{f}([X,Y] \wedge Z) \end{aligned}$$

(we are using that $[X,Z] = [Y,Z] = 0$ since $\mathfrak{I}^2$ is central in $\mathfrak{I}$). As $[\mathfrak{I}^1, \mathfrak{I}^1] = \mathfrak{I}^2$, we see that $\mathtt{f}$ vanishes on all of $\mathfrak{I}^2 \wedge \mathfrak{I}^2 = \mathrm{gr}^4(\bigwedge^2_{\overline{\mathbb{F}}_p} \mathfrak{I})$, and therefore is trivial. This implies that $\partial_3|_{\mathrm{gr}^4(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p))}$ is injective, and consequently

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{-4,6}(\mathfrak{I}, \overline{\mathbb{F}}_p)\right) = \dim_{\overline{\mathbb{F}}_p}\left(h^2(\mathrm{gr}^{-4}(C^\bullet(\mathfrak{I}, \overline{\mathbb{F}}_p)))\right) = 0. \tag{20}$$

Combining equations (17), (18), (19), and (20), we obtain

$$\dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^2(I_1, \overline{\mathbb{F}}_p)\right) \overset{(16)}{\leq} \sum_{i \in \mathbb{Z}} \dim_{\overline{\mathbb{F}}_p}\left(\mathrm{H}^{i,2-i}(\mathfrak{I}, \overline{\mathbb{F}}_p)\right) = (2f^2 - 2f) + 2f = 2f^2.$$

Combining this with the lower bound (10) and equation (15) gives the following.

Theorem 0.4. *Suppose $p \geq 5$ and F is unramified over $\mathbb{Q}_p$ of degree f. We then have $\dim_{\overline{\mathbb{F}}_p}(\mathrm{H}^2(I_1, \overline{\mathbb{F}}_p)) = 2f^2$. Moreover, as a right $\mathcal{H}$-module, we have*

$$\mathrm{H}^2(I_1, \overline{\mathbb{F}}_p) \cong \begin{cases} \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_T}(\varepsilon_{\mathbb{Q}_p} \circ \alpha) & \text{if } F = \mathbb{Q}_p, \\ \left(\mathfrak{ss}_2 \oplus \mathfrak{ss}_{p^2-3} \oplus \mathfrak{ss}_{2p} \oplus \mathfrak{ss}_{p^2-1-2p}\right) \oplus \left(\mathfrak{ss}_{2p-2} \oplus \mathfrak{ss}_{p^2-1-2p+2}\right) & \\ \qquad \oplus \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_T}(\varepsilon_{\mathbb{Q}_{p^2}} \circ \alpha) & \text{if } F = \mathbb{Q}_{p^2}, \\ \left(\bigoplus_{r=0}^{f-1} \mathfrak{ss}_{2p^r} \oplus \mathfrak{ss}_{q-1-2p^r}\right) \oplus \left(\bigoplus_{0 \leq r \neq s \leq f-1} \mathfrak{ss}_{[-2p^r + 2p^s]}\right) & \\ \qquad \oplus \left(\bigoplus_{0 \leq r < s \leq f-1} \mathfrak{ss}_{q-1-2p^r-2p^r} \oplus \mathfrak{ss}_{2p^r+2p^s}\right) & \text{if } f \geq 3. \end{cases} \tag{H^2}$$

The theorem above also gives the structure of $\mathrm{H}^{3f-2}(I_1, \overline{\mathbb{F}}_p)$ by duality.

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